Atoms and photons have triggered the quantum revolution

- The triumphant classical physics at the end of the 19th century
  - Three pillars
    - Mechanics (Newton): celestial bodies motion, machines…
    - Thermodynamics (Boltzmann). The science of heat and engines
    - Electromagnetism (Maxwell). Electricity, magnetism and light
  - Countless success
- Only two ‘little clouds’ (lord Kelvin) at the interfaces…
  - Ether problem (relativity of the velocity of light)
  - Blackbody radiation (light emission by heated bodies)
  - And also
    - Atomic spectra
- …Which turned into a great storm
  - No understanding of microscopic physics in the ‘classical’ frame

The quantum revolution
- 1926-1930
  - An impressive success
    - Atomic structure finally understood
  - Many counterintuitive features
    - Raises fundamental questions on physics
  - A mandatory interpretation
    - Link between the mathematical framework and the experimental reality.
  - A few individuals have changed our understanding of the world

An unprecedented series of successes…
- Huge range of applications
  - From elementary particles and strings…
    - $10^{-35}$ m to $10^{-15}$ m
  - …to cosmological structures…
    - $10^{26}$ m
  - …through atoms, molecules and solids
    - $10^{-10}$ m
- Extremely precise predictions
  - Agreement between theory and experiment over 12 digits!
- A universal theoretical frame
  - All interactions (but gravity) in a single formalism
…countless applications…

- **Atoms and photons**
  - **Lasers** essential for long-haul information transmission in optical fibres i.e. for internet!
  - Ultra-precise atomic clocks (1s over the age of the Universe) are at the heart of the GPS system

- **Solid state physics**
  - **Integrated circuits** rely on our quantum understanding of electrical currents in semiconductors (silicon)

**Nuclear magnetic resonance imaging (MRI)** is a combination of quantum technologies…. Superconducting magnets

Quantum dance of the Hydrogen nuclei in magnetic fields

Data processors for image reconstruction

...countless applications...

A considerable societal and economic impact

- Large part of our GDP (40%) results from quantum technologies
  - Also large part of our lifetime expectancy!
- No information society without the quantum

An astounding example of the impact of curiosity-driven blue-sky research on the long term

- Science needs time
  - One century of quantum physics to shape the society
  - Far beyond the 5 years horizon of science granting systems
- Science needs freedom
  - Investigate exotic problems even though they do not look like a scientific priority at the time
Atoms and light

- Have initiated the quantum revolution
- Continued to play a central role in the development of quantum physics
  - On a conceptual level
    - Problems of diverging energy shifts in vacuum (Lamb shift) and electron magnetic moment (g-2) led to renormalization techniques, modern quantum electrodynamics and, later, to the standard model
  - On an experimental level
    - Development of increasingly sophisticated methods to manipulate atoms with light (or light with atoms)
    - Periodic technological breakthroughs lead to periodic revivals of AMO physics

- Let us discuss a short history of these developments


Ramsey 1949

- Separated oscillatory field method
  - A first example of quantum interferometry
  - A powerful spectroscopic tool
  - Key ingredient in atomic clocks
  - Much more on Ramsey fringes in these lecture

Rabi 1935

- Inducing rf transitions in an atom
  - First manipulation of an atomic spin by radiofrequencies
  - Still a key ingredient in all AMO experiments

Bloch and Purcell 1946

- Nuclear magnetic resonance
  - RF Manipulation of nuclear spins in magnetic fields
  - A key technology for chemical analysis
  - and for medical imaging
Brossel and Bitter 1950

- Double resonance
  - Optical detection of radiofrequency transitions
    - Observation of multiple quanta transitions
    - A harbinger of optical pumping

Kastler and Brossel 1950

- Optical pumping
  - Manipulation of the atoms angular momentum by the angular momentum of light
    - Creation of out-of-equilibrium situations
    - A powerful tool for atomic physics
    - Suggestion to manipulate also the external degrees of freedom
      - Effet lumino-frigorique

Townes, Gordon, Schawlow, Mainman, Javan…1960

- Masers and lasers
  - A technological revolution in AMO physics
    - Infinitely powerful, infinitely sharp...
    - Allowed us also to enter digital communication age

Non linear phenomena… 1960-1970

- Franken, Bloembergen
  - Non linear optics
    - Harmonic generation, frequency mixing...

- Bordé, Hall
  - Saturated absorption
    - Doppler-free spectroscopy

- Grynberg, Cagnac, Chebotaiev
  - Two-photon doppler-free spectroscopy
    - High resolution spectroscopy of narrow atomic lines
    - Toward Hydrogen spectroscopy
1980 More control on individual quantum particles

- Atomic cooling techniques
  - Doppler, molasses, MOT

- Ion traps
  - View individual particles
  - Quantum jumps revealed

- Individual photons
  - Correlations Hong-Ou-Mandel

- Cavity QED
  - One atom and one field mode
  - Spontaneous emission control
  - Much more on that soon

1990 Quantum information and new states of matter

- Quantum entanglement
  - With photons, atoms…
  - Quantum cryptography
  - Quantum teleportation
    - Toward the quantum internet

- Limits of quantum entanglement
  - Decoherence

- Bose-Einstein condensation
  - Textbook example of quantum phase

2000-2020 The success story goes on...

- Frequency combs
  - A revolution in laser metrology
    - Ultra precise clocks

- Attosecond physics
  - Atomic phenomena in real time

- Artificial systems for quantum information
  - Tailored 'atoms'

- New phases of ultra-cold matter
  - Atomic lattices
  - Convergence with solid-state

9 or 10 orders of magnitude gain in 50 years on:

- Clock precision
  - From $10^{-8}$ for quartz clocks to $10^{-18}$ for lattice clocks

- Spectroscopic resolution
  - $10^5$ to $10^{15}$ on Hydrogen spectroscopy

- Atomic detection efficiency
  - From $10^{10}$ atoms to 1 atom

- Atomic temperatures
  - From 1 K to sub-nK

- Laser pulse duration
  - From 10 ps (modelock lasers) to 100 as

- Laser peak intensity
This school

- Interaction of light with cold atoms
  - A key ingredient in modern AMO and quantum optics
    - A prerequisite for any research in this domain
  - A paradigmatic example of fundamental quantum process
    - An opportunity to wonder on the quantum
  - Opens fascinating routes for fundamental physics and applications
    - New states of matter
    - Quantum communication
    - Quantum computing
    - Quantum simulation
    - Quantum metrology

The program

- Jean-Michel Raimond, LKB, SU: *Introductory lecture, Field quantization, Dressed states and Cavity QED*
- Mikhail Baranov, University of Innsbruck: *Quantum gases and superfluidity*
- Hélène Perrin, LPL, CNRS, University Paris 13: *Optical lattices*
- Philippe Verkerk, PhLAM, CNRS, University of Lille: *Laser cooling*
- Ana Asenjo Garcia, Columbia University: *Collective phenomena in light-matter interfaces*
- Antoine Browaeys, Institut d'Optique Graduate School, CNRS, Université Paris Saclay: *Dipole - dipole interaction*
- Robin Kaiser, INPHYNI, CNRS, UCA: *Interaction and disorder*
- Ekkehard Peik, PTB: *Applications: from high precision measurements to metrology*
- Jook Walraven, University of Amsterdam: *Ultracold collisions*

Mikhail Baranov: *Quantum gases and superfluidity*

Hélène Perrin: *Optical lattices* (3 lectures)

- Lecture 1: Band structure in a periodic potential: Bloch functions, energy bands
- Lecture 2: Dynamics in the lattice: time-of-flight, adiabatic switching and band mapping, Bloch oscillations
- Lecture 3: Tight-binding limit: from Wannier functions to the Bose-Hubbard Hamiltonian. Mott transition
Philippe Verkerk: Laser cooling (4 lectures)

- **Lecture 1**: Two-level atoms; light forces
  - Two-level atom
  - Semi-classical description
  - Broad-band limit
  - Force operator
- **Lecture 2**: Doppler cooling and the magneto-optical trap
- **Lecture 3**: Sub-Doppler cooling
- **Lecture 4**: Beyond the simple MOT; non-linear objects

Ana Asenjo-Garcia:
Collective phenomena in light-matter interfaces

- 1- Atom-light interaction as a spin model
  - quantization of the electromagnetic field using Green's functions
  - meaning of Born, Markov, and rotating wave approximations
  - effective spin model for atom-atom interactions
- 2- Atom arrays as light-matter interfaces
  - collective phenomena: super- and sub-radiance
  - physics of subradiant states in 1D and 2D ordered arrays
  - classical vs quantum interference
- 3- Atom-atom interactions in non-conventional baths
  - modifying interactions through nanophotonics
  - waveguide QED: atoms and other quantum emitters close to fibers and photonic
  - crystals
  - applications for quantum information science

Robin Kaiser: Interaction and disorder:
Light scattering by cold atoms / Localization and Cooperative Scattering

- **Lecture 1**: Multiple Scattering of Light in Cold Atoms
  - steady-state results: Ohm's law for photons
  - time dependent scattering: radiation trapping
  - numerical random walk simulations

- **Lecture 2**: Interference Effects in Light Scattering by Cold Atoms
  - coherent backscattering of light by cold atoms
  - numerical simulations with weak localization corrections
  - 'single photon' Dicke super- and subradiance
  - numerical simulations with coupled dipoles

- **Lecture 3**: Anderson Localisation of Light
  - Anderson lattice model
  - effective Hamiltonians approach
  - scalar vs vectorial light: red light for Anderson localization
  - outlook: towards localization of light in cold atoms

Link to Matlab codes:
http://www.kaiserlux.de/coldatoms/LesHouches2019Kaiser.html

Dipole interactions between atoms: QIP and many-body physics

Antoine Browaesys

1. Dipole-dipole interactions between atoms
2. Many-body physics with synthetic matter
3. Quantum information processing

10 μm

Gallery of assembled 2D arrays...

Applications: from high precision measurements to metrology

- High resolution spectroscopy of cold trapped atoms and ions
- Atomic clocks in the microwave and optical domain
- How to measure and quantify uncertainty and instability
- Methods and techniques: Ramsey spectroscopy schemes; laser frequency stabilisation, femtosecond frequency combs
- Frequency comparisons as tests of fundamental principles of physics

Jook Walraven, University of Amsterdam: Ultracold collisions

1. Relative motion of interacting particles I
   - model potentials: range, phase shift, scattering length
2. Relative motion of interacting particles II
   - model potentials: effective range and s-wave resonance
   - generalization to arbitrary short-range potentials
3. Scattering of interacting particles
   - scattering amplitude and cross section
   - distinguishable versus identical particles
4. Scattering of particles with internal structure (atoms)
5. Interaction tuning with magnetic Feshbach resonances

This lecture: Field quantization and CQED

- A basic introduction to quantum atom-field interaction
  - Field quantization
  - Atom-field interaction
  - Cavity quantum electrodynamics

Bibliography

- All quantum optics textbooks
  - In particular:
    - Cohen-Tannoudji and Guery Odelin Advances in atomic physics: an overview, World Scientific 2012
    - Schleich, Quantum optics in phase space, Wiley 2000;
    - Vogel, Welsch and Wallentowitz, Quantum optics an introduction, Wiley 2001
    - Meystre and Sargent Elements of quantum optics, Springer 1999
    - Barnett and Radmore Methods in theoretical quantum optics, OUP, 1997
    - Scully and Zubairy Quantum optics, 1997
Bibliography

- The notes of « atoms and photons » lecture at M2 ICFP (ENS)
  - [http://www.lkb.upmc.fr/cqed/teaching/teachingjmr/](http://www.lkb.upmc.fr/cqed/teaching/teachingjmr/)

- S. Haroche and J.M. Raimond
  - Exploring the quantum:
    - atoms, cavities and photons
  - Oxford Univ. Press 2006
  - And many references therein

Let us start and quantize the field
Field quantization and cavity QED
Field quantization

J.M. Raimond

September 23, 2019

Outline

1. Field eigenmodes

2. Field quantization, Fock states

3. Other field quantum states

4. Field relaxation
Outline

1 Field eigenmodes
2 Field quantization, Fock states
3 Other field quantum states
4 Field relaxation

Objective

To quantify the field, we must identify a set of orthogonal modes, the relevant dynamical variables and quantify them according to the ‘canonical’ quantization procedure. The main technical difficulty in field quantization is thus a classical electromagnetism calculation.

Field eigenmodes

Positive frequency fields
Temporal Fourier transform of electric field

\[ E(r, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{E}(r, \omega) e^{-i\omega t} d\omega \]  

(1)

Since \( E \) is a real field,

\[ \tilde{E}^*(r, \omega) = \tilde{E}(r, -\omega) \]  

(2)

Positive frequency field

\[ E^+(r, t) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \tilde{E}(r, \omega) e^{-i\omega t} d\omega \]  

(3)

Negative frequency field

\[ E^-(r, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \tilde{E}(r, \omega) e^{-i\omega t} d\omega = (E^+(r, t))^* \]  

(4)

Hence

\[ E(r, t) = E^+(r, t) + E^-(r, t) \]  

(5)
Eigenmodes

Eigenmodes basis

Note: we will use a slightly different normalization condition for a physical quantization box (cavity QED), based on the geometry of the single mode of interest.

Expand the positive frequency field on this basis

\[ \mathbf{E}^+(\mathbf{r}, t) = \sum_{\ell} \mathcal{E}_\ell(t) \mathbf{f}_\ell(\mathbf{r}) \]  

\[ \mathcal{E}_\ell(t) = \frac{1}{V} \int \mathbf{E}^+(\mathbf{r}, t) \cdot \mathbf{f}^*_\ell(\mathbf{r}) \, d^3r \]  

The amplitude is obviously a harmonic function of time

\[ \mathcal{E}_\ell(t) = \mathcal{E}_\ell(0) e^{-i\omega_\ell t} \]  

Finally

\[ \mathbf{E}^+(\mathbf{r}, t) = \sum_{\ell} \mathcal{E}_\ell(0) e^{-i\omega_\ell t} \mathbf{f}_\ell(\mathbf{r}) \]

Normal variables

Potential vector

Choose a simple set of dynamical variables. The potential vector \( \mathbf{A} \) is divergence-free in the Coulomb gauge \( \nabla \cdot \mathbf{A} = 0 \) and \( \mathbf{E} = -\partial \mathbf{A} / \partial t \).

\( \mathbf{A} \) can be expanded on the same basis as \( \mathbf{E} \) (same limiting conditions)

\[ \mathbf{A}^+(\mathbf{r}, t) = \sum_{\ell} \mathcal{A}_\ell(t) \mathbf{f}_\ell(\mathbf{r}) \]  

Choose \( \mathcal{A}_\ell(t) \) (harmonic functions of time) as the normal variables and separate real and imaginary parts

\[ \mathcal{A}_\ell(t) = \mathcal{A}_\ell(0) e^{-i\omega_\ell t} = \frac{1}{2\sqrt{\epsilon_0 \omega_\ell}} \left[ \chi_\ell(t) + i \rho_\ell(t) \right] , \]

where we have introduced a normalization factor simplifying the final form of the field energy/hamiltonian in terms of the normal variables.

Note that \( \chi_\ell \) and \( \rho_\ell \) have the dimension of the square root of an action...
Field eigenmodes

Normal variables
All fields

From $E^+ = -\partial A^+/\partial t$

$E_\ell(t) = \frac{dA_\ell}{dt} = i\omega_\ell A_\ell$  \hspace{1cm} (19)

and hence

$E^+(r, t) = \sum_\ell i\omega_\ell A_\ell(t)f_\ell(r)$  \hspace{1cm} (20)

Magnetic field:

$B^+(r, t) = \sum_\ell A_\ell(t)h_\ell(r)$  \hspace{1cm} (21)

where

$h_\ell(r) = \nabla \times f_\ell(r)$  \hspace{1cm} (22)

Field energy

The total field energy

$H = \frac{\varepsilon_0}{2} \int E^2 + \frac{1}{2\mu_0} \int B^2$  \hspace{1cm} (23)

must be written in terms of real fields

$E = 2\text{Re} E^+ = 2\text{Re} \sum_\ell i\omega_\ell A_\ell f_\ell$  \hspace{1cm} (24)

Taking into account the modes orthogonality conditions

$H = \sum_\ell H_\ell$  \hspace{1cm} (25)

Remains to evaluate energy of one given mode. Drop index $\ell$ for the time being.

Field energy

Electric energy

Real field

$E = i\omega [A f - A^* f^*]$  \hspace{1cm} (26)

or

$E = -\sqrt{\frac{\omega}{\varepsilon_0 V}} [xf'' + pf']$  \hspace{1cm} (27)

with

$f = f' + if''$  \hspace{1cm} (28)

$H_e = \frac{\omega}{2V} \left[ x^2 \int (f')^2 + p^2 \int (f')^2 + 2xp \int f' \cdot f'' \right]$  \hspace{1cm} (29)

Magnetic energy

With

$B = A h + A^* h^* = \frac{1}{\sqrt{\omega_0 V}} [x h' - 2ph'']$  \hspace{1cm} (30)

we get

$H_b = \frac{c^2}{2\omega V} \left[ x^2 \int (h')^2 + p^2 \int (h')^2 - 2xp \int h' \cdot h'' \right]$  \hspace{1cm} (31)

Similar, but not obviously equal, to the electric energy.
Field energy
Comparing the energies
Let us start with the integral of $(h^0)^2$, with $h = \nabla \times f$. Using

$$\nabla \cdot (a \times b) = b \cdot (\nabla \times a) - a \cdot (\nabla \times b)$$  \hspace{1cm} (32)$$

we can write

$$\nabla \cdot [f^0 \times (\nabla \times f')] = (\nabla \times f')^2 - f' \cdot (\nabla \times \nabla \times f')$$  \hspace{1cm} (33)$$

Using that these fields are divergence-free and with Helmholtz equation:

$$\nabla \cdot [f^0 \times (\nabla \times f')] = (h^0)^2$$  \hspace{1cm} (34)$$

Integrating over space:

$$\int (h^0)^2 = \frac{\omega^2}{c^2} \int (f')^2$$  \hspace{1cm} (35)$$

Similarly

$$\int (h''')^2 = \frac{\omega^2}{c^2} \int (f'')^2$$  \hspace{1cm} (36)$$

The total energy of the radiation field is thus

$$H = \sum \omega \left[ x^2 + p^2 \right]$$  \hspace{1cm} (45)$$

Field energy
Final result for a single mode
We get finally

$$H = \frac{\omega}{2} \left[ x^2 + p^2 \right]$$  \hspace{1cm} (42)$$

the Hamilton function of a one-dimensional harmonic oscillator.
Note that $x$ and $p$ are canonically conjugate variables since

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = \omega p$$  \hspace{1cm} (43)$$

as expected for an exp($-i\omega t$) dependence of $A(t)$. With $p$, we get

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} \hspace{1cm} \text{and} \hspace{1cm} \frac{dp}{dt} = -\frac{\partial H}{\partial x}$$  \hspace{1cm} (44)$$

as required for canonically conjugate variables, suitable for the quantization procedure.
Field eigenmodes

Field momentum

Total momentum

Density of momentum proportional to the Poynting vector

\[ g = \frac{\mathbf{P}}{c^2} \quad \text{with} \quad \mathbf{P} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \quad (46) \]

The plane wave mode basis is most convenient to describe the momentum

\[ \mathbf{E}^+(\mathbf{r}, t) = \sum_\ell \mathbf{E}^+_{\ell}(t) e^{i k_{\ell} \cdot \mathbf{r}} \quad (47) \]

and

\[ \mathbf{B}^+(\mathbf{r}, t) = \sum_\ell \mathbf{B}^+_{\ell}(t) e^{i k_{\ell} \cdot \mathbf{r}} \quad (48) \]

After a painful calculation

Total momentum

\[ \mathbf{P} = \frac{1}{2} \sum_\ell |x_\ell + i p_\ell|^2 k_\ell \quad (51) \]

with a clear interpretation.

Field quantization

The field is a collection of independent harmonic oscillators quantified independently, using the Dirac approach.

The conjugate classical variables \( x \) and \( p \) (we drop the index \( \ell \) in this section) are replaced by two operators \( X \) and \( P \) (position and momentum operators, with the dimension of the square root of an action) acting in an infinite dimension Hilbert space, with the commutation rule:

\[ [X, P] = i\hbar \quad (52) \]

We define the reduced quadratures by:

\[ X_0 = \frac{X}{\sqrt{2\hbar}} \quad \text{and} \quad P_0 = \frac{P}{\sqrt{2\hbar}} \quad (53) \]

With these definitions

\[ [X_0, P_0] = \frac{i}{2} \quad (54) \]

We then define the creation and annihilation operators

\[ a = X_0 + iP_0, \quad a^\dagger = X_0 - iP_0, \quad X_0 = \frac{a + a^\dagger}{2}, \quad P_0 = i \frac{a^\dagger - a}{2} \quad (55) \]

and we get

\[ [a, a^\dagger] = 1 \quad (56) \]
Field quantization

Hamiltonian

From the classical field energy, we get the quantum field Hamiltonian

\[ H = \frac{\omega}{2} (X^2 + P^2) = \hbar \omega (X_0^2 + P_0^2) \]  

or

\[ H = \frac{\hbar \omega}{4} \left( (a + a^\dagger)^2 - (a^\dagger - a)^2 \right) \]  

Normal order Hamiltonian

\[ H = \hbar \omega \left( a^\dagger a + \frac{1}{2} \right) \]  

whose diagonalization is described in all textbooks.

Field quantization

Fock states

|n⟩ are the ‘photon number states’ with the orthogonality relation

\[ \langle n | p \rangle = \delta_{n,p} \]  

Annihilation of photons:

\[ a |n⟩ = \sqrt{n} |n-1⟩ \]  

with

\[ a |0⟩ = 0 \]  

Similarly

\[ a^\dagger |n⟩ = \sqrt{n+1} |n+1⟩ \]  

Hence

\[ |n⟩ = \frac{(a^\dagger)^n}{\sqrt{n!}} |0⟩ \]  

Photon number distribution

\[ p_n = |c_n|^2 \]  

Mean and variance of photon number

\[ \bar{n} = \sum_n n p_n \quad \Delta N^2 = \langle N^2 \rangle - \langle N \rangle^2 = \sum_n (n - \bar{n})^2 p_n \]  

Statistical mixtures

\[ \rho = \sum_{n,p} \rho_{np} |n⟩ ⟨p| \]  

Photon number distribution

\[ \rho_{nn} = \rho_n \]  

Field quantization

Number operator

\[ N = a^\dagger a \]  

Commutation relations:

\[ [N,a] = -a \quad \text{and} \quad [N,a^\dagger] = a^\dagger \]  

Eigenvectors: all positive integers, with non-degenerate eigenstates

\[ N |n⟩ = n |n⟩ \]  

Hence, the eigenvalues are

\[ E_n = \left( n + \frac{1}{2} \right) \hbar \omega \]  

Ground state: ‘vacuum’, |0⟩, energy \( \hbar \omega / 2 \)
Fock states

Wavefunctions

Basis of eigenstates of the quadratures:

\[ X_0 |x\rangle = x |x\rangle \quad \text{and} \quad P_0 |p\rangle = p |p\rangle \quad (73) \]

Wavefunctions:

\[ \Psi(x) = (x |\Psi) \quad (74) \]

For the vacuum:

\[ \Psi_0(x) = \left(\frac{2}{\pi}\right)^{1/4} e^{-x^2} \quad (75) \]

Also in the |p\rangle representation:

\[ \tilde{\Psi}_0(p) = \left(\frac{2}{\pi}\right)^{1/4} e^{-p^2} \quad (76) \]

Suggests a pictorial representation of the vacuum as a small circle in phase plane.

Fock states

Wavefunctions

For the Fock state |n\rangle:

\[ \Psi_n(x) = \left(\frac{2}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} e^{-x^2} H_n(x\sqrt{2}) \quad (77) \]

where \( H_n \) is the \( n \)th Hermite polynomial defined by

\[ H_n(u) = (-1)^n e^{u^2} \frac{d^n}{du^n} e^{-u^2} \quad (78) \]

These wavefunctions have \( n \) nodes and a parity \((-1)^n\)

Field operators

All modes

\[ H |n_1, \ldots, n_\ell \ldots\rangle = E_n |n_1, \ldots, n_\ell \ldots\rangle \quad (79) \]

with

\[ E_n = \sum_\ell \left( n_\ell \hbar \omega_\ell + \frac{\hbar \omega_\ell}{2} \right) \quad (80) \]

and

\[ |n_1, \ldots, n_\ell \ldots\rangle = \prod_\ell \frac{(a_\ell^\dagger)^{n_\ell}}{\sqrt{n_\ell!}} |0\rangle \quad (81) \]

Note that the vacuum state has an infinite energy!
Field quantization, Fock states

Field operators

Vector potential operator

Classical normal variables:

\[ A_\ell = \frac{1}{2\sqrt{\varepsilon_0 \omega V}}(\chi_\ell + ip_\ell) \]  

(82)

Corresponding quantum operators

\[ A_\ell = \frac{1}{2\sqrt{\varepsilon_0 \omega V}}(X_\ell + iP_\ell) = \sqrt{\frac{\hbar}{2\varepsilon_0 \omega V}} a_\ell \]  

(83)

Positive frequency vector potential

\[ A_\ell^+ = \sum_\ell \sqrt{\frac{\hbar}{2\varepsilon_0 \omega V}} a_\ell f_\ell(r) \]  

(84)

Hermitian vector potential:

\[ A(r) = \sum_\ell \sqrt{\frac{\hbar}{2\varepsilon_0 \omega V}} \left( a_\ell f_\ell(r) + a_\ell^* f_\ell^*(r) \right) \]  

(85)

Field operators

Electric and magnetic field operators

The hermitian electric field is similarly:

\[ E(r) = i \sum_\ell \mathcal{E}_\ell \left( a_\ell f_\ell(r) - a_\ell^* f_\ell^*(r) \right) \]  

(86)

where we define the ‘field per photon in mode \( \ell \)’ by

\[ \mathcal{E}_\ell = \sqrt{\frac{\hbar \omega_\ell}{2\varepsilon_0 V}} \]  

(87)

Similarly,

\[ B(r) = \sum_\ell \sqrt{\frac{\hbar}{2\varepsilon_0 \omega V}} \left( a_\ell h_\ell(r) + a_\ell^* h_\ell^*(r) \right) \]  

(88)

with \( h_\ell = \nabla \times f_\ell \)

Plane wave mode basis

\[ A^+_\ell(r) = \sum_\ell \sqrt{\frac{\hbar}{2\varepsilon_0 \omega V}} a_\ell e^{i k_\ell \cdot r} \]  

(89)

\[ E^+_\ell(r) = i \sum_\ell \mathcal{E}_\ell a_\ell e^{i k_\ell \cdot r} \]  

(90)

\[ B^+_\ell(r) = \sum_\ell \sqrt{\frac{\hbar}{2\varepsilon_0 \omega V}} a_\ell (ik_\ell \times \epsilon_\ell) e^{i k_\ell \cdot r} \]  

(91)

Evolution of annihilation operator

\[ i\hbar \frac{d a_H}{dt} = [a_H, H] \quad \text{i.e.} \quad \frac{d a_H}{dt} = -i\omega a_H \]  

(92)

whose immediate solution is

\[ a_H(t) = a_H(0)e^{-i\omega t} = ae^{-i\omega t} \]  

(93)

Same harmonic evolution as the classical normal variables.
Field operators

Momentum

Total momentum by replacing $|x + ip|^2$ in the classical expression by $(x - ip)(x + ip)$ and $x + ip$ by $a^\dagger a$:

$$P = \sum_\ell \hbar k_\ell a^\dagger_\ell a_\ell$$  \hspace{1cm} (94)

Fock states

Non-classicality of Fock states

Fock states are very non-classical
- A large energy
- Zero average fields and potentials since $\langle n | a | n \rangle = 0$

Can we find more intuitive field states? Yes: Coherent states.

Coherent states

Displacement operator

A unitary defined by:

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$$  \hspace{1cm} (95)

where $\alpha$ is an arbitrary complex amplitude

$$\alpha = \alpha' + i\alpha''$$  \hspace{1cm} (96)

$$D(\alpha)^\dagger D(\alpha) = 1$$  \hspace{1cm} (97)

and

$$D(\alpha)^\dagger = D(-\alpha)$$  \hspace{1cm} (98)

Coherent states

Displacement operator

An equivalent expression

$$D(\alpha) = e^{2i\alpha''x_0 - 2i\alpha'p_0}$$  \hspace{1cm} (99)

Using the Glauber relation

$$e^A e^B = e^{A+B+e[A,B]/2}$$  \hspace{1cm} (100)

valid when

$$[A, [A, B]] = [B, [A, B]] = 0$$  \hspace{1cm} (101)

We get

$$D(\alpha) = e^{-i\alpha'\alpha''} e^{2i\alpha''x_0} e^{-2i\alpha'p_0}$$  \hspace{1cm} (102)

a product of displacement operators:

$$e^{-2i\alpha'p_0} |x\rangle = |x + \alpha'\rangle$$  \hspace{1cm} (103)

$$e^{2i\alpha''x_0} |p\rangle = |p + \alpha''\rangle$$  \hspace{1cm} (104)
Coherent states

Combination of displacements

Using Glauber

\[ D(\alpha)D(\beta) = e^{(\alpha\beta^* - \alpha^*\beta)/2}D(\alpha + \beta) \]  

(105)

Note that

\[ \Phi = (\alpha\beta^* - \alpha^*\beta)/2i = \frac{\alpha''\beta' - \alpha'\beta''}{2} \]  

(106)

is the surface of the triangle with sides \( \alpha \) and \( \beta \).

The coherent states are defined as displaced vacuum states

\[ |\alpha\rangle = D(\alpha)|0\rangle. \]  

(109)

Note that \( |0\rangle \) is a coherent state.

Wavefunction of a coherent state in the \( X_0 \) representation:

\[ \Psi_{\alpha}(x) \propto e^{-(x-\alpha)^2} \]  

(110)

and in the \( P_0 \) representation:

\[ \tilde{\Psi}_{\alpha}(p) \propto e^{-(p-\alpha')^2} \]  

(111)
Coherent states

Properties

• Right-eigenstates of the annihilation operator

\[ a | \alpha \rangle = a D(\alpha) | 0 \rangle = D(\alpha) D(-\alpha) a D(\alpha) | 0 \rangle = D(\alpha)(a+\alpha \mathbb{1}) | 0 \rangle = \alpha | \alpha \rangle \]

(112)

since \( a | 0 \rangle = 0 \). Hence

\[ \langle \alpha | a | \alpha \rangle = \alpha \quad \text{and} \quad \langle \alpha | a^\dagger | \alpha \rangle = \alpha^* \]

(113)

• Field operators have nonzero eigenvalues in the coherent states:

\[ \langle E \rangle = i \mathcal{E} [f(r)\alpha - f^*(r)\alpha^*] \]

(114)

\[ \langle A \rangle = \frac{\mathcal{E}}{\omega} [f(r)\alpha + f^*(r)\alpha^*] \]

(115)

Coherent states

Properties

• Average photon number

\[ \bar{n} = \langle \alpha | a^\dagger a | \alpha \rangle = | \alpha |^2 \]

(116)

• Photon number variance. Using \( N^2 = a^\dagger a a^\dagger a = (a^\dagger)^2 a^2 + a^\dagger a \)

\[ \langle N^2 \rangle = | \alpha |^4 + | \alpha |^2 \]

(117)

and

\[ \Delta N^2 = | \alpha |^2 = \bar{n} \]

(118)

\[ \frac{\Delta N}{\bar{n}} = \frac{1}{\sqrt{\bar{n}}} \]

(119)

Coherent states

Properties

• Expansion on the Fock state basis

\[ D(\alpha) = e^{-|\alpha|^2/2} e^{a\alpha^*} \]

(120)

with \( a | 0 \rangle = 0 \):

\[ | \alpha \rangle = e^{-|\alpha|^2/2} e^{a\alpha^*} | 0 \rangle \]

(121)

Expand exponential:

\[ | \alpha \rangle = \sum_n c_n | n \rangle \]

(122)

with

\[ c_n = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} \]

(123)

Coherent states

Properties

• Photon number distribution

\[ \rho_n = e^{-|\alpha|^2} | \alpha |^{2n} n! = \frac{e^{-\bar{n}} \bar{n}^n}{n!} \]

(124)

For large average photon numbers \( \rho_n \propto e^{-(n-\bar{n})^2/\bar{n}} \)

• Scalar product of coherent states

\[ \langle \alpha | \beta \rangle = e^{-|\alpha|^2+|\beta|^2}/2 \sum_{n,p} (\alpha^*)^n \beta^p \sqrt{n!p!} \langle n | p \rangle \]

\[ = e^{-|\alpha|^2+|\beta|^2}/2 e^{\alpha^*\beta} \]

(125)

\[ | \langle \alpha | \beta \rangle |^2 = e^{-|\alpha-\beta|^2} \]

(126)

• Overcomplete basis (expansion of arbitrary state not unique)

\[ \mathbb{1} = \frac{1}{\pi} \int d^2 \alpha | \alpha \rangle \langle \alpha | \]

(127)
### Coherent states

#### Properties

- **Evolution**
  
  \[ |\Psi(t)\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \]

  \[ |\Psi(t)\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-i\omega t/2} |n\rangle \]

  Evolution of the amplitude is the same as in classical physics

  \[ \alpha(t) = \alpha(0)e^{-i\omega t} \]

  The Husimi-Q representation

  \[ Q[|\alpha\rangle] = \frac{1}{\pi} \text{Tr} [\rho |\alpha\rangle \langle \alpha|] = \frac{1}{\pi} \langle \alpha| \rho |\alpha\rangle = \frac{1}{\pi} \text{Tr}[[0] |\alpha\rangle \langle \alpha| D(-\alpha) |0\rangle] \]

  The Q distribution is positive, bounded by 1/\pi and normalized (\int d^2\alpha Q(\alpha) = 1).

  A few states:

  - Coherent state |\beta\rangle
  
    \[ Q[|\beta\rangle] = \frac{1}{\pi} |\beta| \langle \beta| \langle \beta| = \frac{1}{\pi} e^{-|\beta|^2} \]

  - Fock state |n\rangle
  
    \[ Q[|n\rangle |n\rangle] = \frac{1}{\pi} |n|^{2n} e^{-|n|^2} \]
Other field quantum states

Phase space representations

The Husimi-$Q$ representation

- Cat state
  \[ |\psi_{\text{cat}}^{\pm}\rangle = \frac{1}{\sqrt{N_{\pm}}}(|\beta\rangle \pm |{-}\beta\rangle) \]  
  \[ \text{where:} \quad N_{\pm} = 2 \left(1 \pm e^{-2|\beta|^2}\right) \]  
  \[ Q^{\text{cat,}\pm}(\alpha) = \frac{1}{\pi N_{\pm}} \left[ e^{-|\alpha-\beta|^2} + e^{-|\alpha+\beta|^2} \pm 2e^{-|(\alpha+\beta)|^2} \cos(2\beta\alpha') \right] \]

(a) Coherent state $|\beta\rangle$, with $\beta = \sqrt{5}$. (b) Five-photon Fock state. (c) Schrödinger cat state, superposition of two coherent fields $|\pm\beta\rangle$, with $\beta = \sqrt{5}$. (d) Statistical mixture of the same coherent components.

Phase space representations

The Wigner function

Definition

\[ W(x, p) = \frac{2}{\pi} \text{Tr}[D(-\alpha)\rho D(\alpha)\mathcal{P}] \]  

where the unitary parity operator $\mathcal{P}$ is defined by

\[ \mathcal{P}|x\rangle = |-x\rangle ; \quad \mathcal{P}|p\rangle = |-p\rangle ; \quad \mathcal{P}|n\rangle = (-1)^n |n\rangle ; \quad \mathcal{P} = e^{i\pi a^2} \]  

The modulus of its average is lower than one. Thus

\[ -2/\pi \leq W(\alpha) \leq 2/\pi \]

Marginals of the Wigner distribution:

\[ P(x) = \langle x| \rho |x\rangle = \int dp \ W(x, p) \]  

\[ P(p) = \langle p| \rho |p\rangle = \int dx \ W(x, p) \]

The average of any operator can be directly obtained from the Wigner function

\[ \langle O \rangle = \int dx dp \ W(x, p) o_{\alpha}(x, p) \]  

where $o_{\alpha}$ is the symmetrized form of the operator $O$ in terms of the field quadratures.
The Wigner function
Phase space representations

A few ‘non-classical’ states

- Coherent state

\[ W^{[\beta] } (\alpha) = \frac{2}{\pi} e^{-2|\beta - \alpha|^2} \]  

(148)

- Thermal field

\[ W^{[\alpha_0]} (\alpha) = \frac{2}{\pi} \frac{1}{2\hbar + 1} e^{-2|\alpha|^2 / (2\hbar + 1)} \]  

(149)

- Squeezed vacuum \( S(\xi) |0\) with

\[ S(\xi) = e^{(\xi^* x^2 - \xi^* x^2)/2} \]  

(150)

with

\[ \Delta X_0 = \frac{1}{2} e^{-\xi} \quad \text{and} \quad \Delta P_0 = \frac{1}{2} e^\xi \]  

(151)

\[ W^{[\alpha, \xi]} (x, p) = \frac{2}{\pi} e^{-2 \exp(2\xi)x^2} e^{-2 \exp(-2\xi)x^2} \]  

(152)

(a) Vacuum state. (b) Coherent state with \( \beta = \sqrt{5} \). (c) Thermal field with \( \hbar = 1 \) photon on the average. (d) A squeezed vacuum state, with a squeezing parameter \( \xi = 0.5 \).

- Fock state

\[ W^{[n]} (n) = \frac{2}{\pi} (-1)^n e^{-2|\alpha|^2} L_n (4|\alpha|^2) \]  

(153)

with

\[ W^{[n]} (n) (0) = \frac{2}{\pi} (-1)^n \]  

(154)

\[ W^{[1]} (1) (\alpha) = - \frac{2}{\pi} (1 - 4|\alpha|^2) e^{-2|\alpha|^2} \]  

(155)

- Cat state

\[ W^{[\text{cat}, \pm]} (\alpha) = \frac{1}{\pi (1 \pm e^{-2|\beta|^2})} \left[ e^{-2|\alpha - \beta|^2} + e^{-2|\alpha + \beta|^2} \right] + 2 e^{-2|\alpha|^2} \cos (4\alpha^* \beta) \]  

(156)

Wigner function of a five-photon Fock state.
Other field quantum states

Phase space representations
The Wigner function

Wigner functions of even (a) and odd (b) 10-photon $\pi$-phase cats.

\begin{itemize}
  \item Quantum system $S$ (the field here, reduced to a single mode for clarity) coupled to an environment $E$. Jointly in a pure state $|\Psi_{SE}\rangle$.
  \item We are interested only in $\rho_S$, obtained by tracing the projector $|\Psi_{SE}\rangle \langle \Psi_{SE}|$ over the environment (the state of the environment is forever inaccessible).
  \item We seek an evolution equation for $\rho_S$ alone.
\end{itemize}

Field relaxation
System and environment

- Transformation of the system’s density matrix during a short time interval
  \[ \rho(t) \rightarrow \rho(t + \tau) \] (157)

- $\tau \gg \tau_c$, correlation time of the reservoir observables, so that there are no coherent effects in the system-reservoir interaction

- This transformation is a ‘quantum map’
  \[ \mathcal{L}(\rho(t)) = \rho(T + \tau) \] (158)

Mathematical properties of a proper quantum map:
- Linear operation, i.e. a super-operator in a space of dimension $N_S^2$ ($N_S$ system’s Hilbert space dimension).
- Preserve unit trace and positivity (a density operator does not have any negative eigenvalue).
- “Completely positive”. If, at a time $t$, $S$ entangled with $S'$, $\mathcal{L}$ acting on $S$ alone leads to a completely positive density operator for the joint state of $S$ and $S'$ (not all maps are completely positive e.g. partial transpose).
Field relaxation

Kraus operators

Any completely positive map can be written as

$$\mathcal{L}(\rho) = \sum_{\mu} M_\mu \rho M^\dagger_\mu$$  \hspace{1cm} (159)

with the normalization condition

$$\sum_{\mu} M^\dagger_\mu M_\mu = 1$$  \hspace{1cm} (160)

There are at most $$N^2$$ ‘Kraus’ operators $$M_\mu$$, which are not uniquely defined (same map when mixing the $$M_\mu$$ by a linear unitary matrix $$V$$:

$$M'_\mu = V_\mu V^\dagger_\nu M_\nu.$$  \hspace{1cm} (165)

Lindblad equation

Kraus representation and differential representation of the map

$$\rho(t + \tau) = \sum_{\mu} M_\mu \rho(t) M^\dagger_\mu \approx \rho(t) + \frac{d\rho}{dt} \tau$$  \hspace{1cm} (164)

- Environment unaffected by the system: the $$M_\mu$$'s do not depend upon time $$t$$.
- They, however, depend clearly upon the small time interval $$\tau$$.
- One and only one of the $$M_\mu$$'s is thus of the order of unity and all others must then be of order $$\sqrt{\tau}$$.

$$M_0 = 1 - iK\tau$$  \hspace{1cm} (165)

$$M_\mu = \sqrt{\tau}L_\mu \quad \text{for} \quad \mu \neq 0$$  \hspace{1cm} (166)

Field relaxation

Kraus operators

Fit in this representation:

- Hamiltonian evolution

$$\rho(t + \tau) = U(\tau) \rho U^\dagger(\tau)$$  \hspace{1cm} (161)

- ‘unread’ generalized measurement

$$\rho \rightarrow \sum_{\mu} O_\mu \rho O^\dagger_\mu$$  \hspace{1cm} (162)

but not a measurement whose result $$\mu$$ is known

$$\rho \rightarrow \frac{O_\mu \rho O^\dagger_\mu}{\text{Tr}(O_\mu \rho O^\dagger_\mu)}$$  \hspace{1cm} (163)

(non-linear normalization term in the denominator)

Lindblad equation

$$K$$, having no particular properties, can be split in hermitian and anti-hermitian parts:

$$K = \frac{H}{\hbar} - iJ$$,  \hspace{1cm} (167)

where

$$H = \frac{\hbar}{2} (K + K^\dagger)$$  \hspace{1cm} (168)

$$J = \frac{i}{2} (K - K^\dagger)$$  \hspace{1cm} (169)

are both hermitian.

$$M_0 = 1 - \frac{i\tau}{\hbar} H - J\tau$$  \hspace{1cm} (170)
Field relaxation

Lindblad equation

Thus

\[ M_0 \rho M_0^\dagger = \rho - \frac{i}{\hbar} [H, \rho] + \tau [J, \rho]_+ \]  

(171)

where \([J, \rho]_+ = J\rho + \rho J\) is an anti-commutator.

\[ M_0^\dagger M_0 = 1 - 2J\tau \] and thus, by normalization since \( \sum_\mu M_\mu^\dagger M_\mu = 1 \)  

(172)

\[ J = \frac{1}{2} \sum_{\mu \neq 0} L_\mu^\dagger L_\mu \]  

(173)

"Lindblad form" of the master equation

\[ \frac{d\rho}{dt} = -\frac{i}{\hbar} [H, \rho] + \sum_{\mu \neq 0} \left( L_\mu \rho L_\mu^\dagger - \frac{1}{2} L_\mu^\dagger L_\mu \rho - \frac{1}{2} \rho L_\mu^\dagger L_\mu \right) \]  

(174)

Field relaxation

Quantum jumps

Consider a single time interval \( \tau \) in the simple situation where the initial state is pure \( \rho(0) = |\Psi\rangle \langle \Psi| \), with no Hamiltonian evolution. Then, omitting normalization:

\[ \rho(\tau) = |\Psi\rangle \langle \Psi| + \tau \sum_{\mu} \langle L_\mu | \Psi \rangle \left( \langle \Psi | L_\mu^\dagger \right) \]  

(175)

- Density matrix at time \( \tau \) is a statistical mixture of the initial pure state (with a large probability of order 1) and of projectors on the states \( L_\mu | \Psi \rangle \).
- The \( L_\mu \)'s are 'jump operators' which describe a random (possibly large) evolution of the system which suddenly (at the time scale of the evolution) changes under the influence of the environment.
- Intuitive picture of quantum jumps for an atom emitting a single photon

The quantum jump operators are not defined unambiguously. Again, the same master equation can be recovered from different sets of \( M_\mu \)'s (or \( L_\mu \)'s) linked together by a unitary transformation matrix. Different choices correspond to different 'unravelings' of the master equation.

In some situations, the quantum jumps have a direct physical meaning, e.g. emitting atom completely surrounded by a photo-detector array. The quantum jump then corresponds to a click of one detector. Different unravelings may then correspond to different ways of monitoring the environment, in this case to different detectors (photon counters, homodyne receivers...)

In other situations, the quantum jumps are an abstract representation of the system+environment evolution.

Field relaxation

Quantum trajectories

- Even when the environment is not explicitly monitored, one may imagine that it is done. We then imagine we have full information about which quantum jump occurs when.
- The system is thus, at any time, in a pure state, which undergoes a stochastic trajectory in the Hilbert space, made up of continuous Hamiltonian evolutions interleaved with sudden quantum jumps.
- However, since we only imagine the information is available, we should describe the evolution of the density operator by averaging the system evolution over all possible trajectories.
- This picture leads to the Monte Carlo trajectory algorithm.
Field relaxation
Quantum Monte Carlo trajectories

- Initialize the state (randomly chosen eigenstate $|\Psi\rangle$ of $\rho$)
- For each time interval $\tau$, evolve $|\Psi\rangle$ according to:
  - Compute $\rho_\mu = \tau (|\Psi\rangle L^\dagger \mu |\Psi\rangle)$ and $\rho_0 = 1 - \sum_{\mu \neq 0} \rho_\mu$.
  - Use a (good) random number generator to decide upon the result of the measurement of $B$.
  - If the result of the measurement is $\mu = 0$, evolve $|\Psi\rangle$ by:
    $|\Psi\rangle \to \frac{L_\mu}{\sqrt{\langle \Psi | L^\dagger \mu |\Psi\rangle}} |\Psi\rangle = \frac{L_\mu}{\sqrt{\rho_\mu}} |\Psi\rangle \quad (177)$
  - If the result of the measurement is $\mu \neq 0$, evolve $|\Psi\rangle$ by:
    $|\Psi\rangle \to \frac{1}{iH\tau / \hbar - J^\dagger \mu} |\Psi\rangle \quad (176)$
- Repeat the procedure for many trajectories
- Average the projectors $\langle \Psi(t) | \langle \Psi(t) |$.

---

Field relaxation
Jump operators

- $L_-$ (or $\sqrt{\kappa_-} a$): loss of a photon in the environment (even when $T = 0$)
- $L_+ = \sqrt{\kappa_+} a^\dagger$: creation of a thermal excitation

Jump rates linked to the temperature of the environment

$$\kappa_+ = \kappa_- e^{-\hbar \omega / k_B T} \quad (178)$$

Using

$$n_{th} = \frac{1}{e^{\hbar \omega / k_B T} - 1} \quad (179)$$

we get

$$\frac{\kappa_-}{\kappa_+} = \frac{1 + n_{th}}{n_{th}} \quad (180)$$

and write

$$\kappa_- = \kappa (1 + n_{th}); \quad \kappa_+ = \kappa n_{th} \quad (181)$$

---

Field relaxation
Lindblad equation

$$\frac{d\rho}{dt} = -i\omega_c [a^\dagger a, \rho] - \frac{\kappa (1 + n_{th})}{2} \left( a^\dagger a \rho + \rho a a^\dagger - 2 a^\dagger a \rho a^\dagger \right) - \frac{\kappa n_{th}}{2} (aa^\dagger \rho + \rho aa^\dagger - 2a^\dagger a \rho) \quad (182)$$

where we have discarded the vacuum energy. Note that all of the Hamiltonian part can be removed by an interaction representation (relaxation terms unchanged). For the photon number distribution:

$$\frac{dp(n)}{dt} = \kappa (1 + n_{th}) (n + 1) p(n + 1) + \kappa n_{th} np(n - 1) - \kappa (1 + n_{th}) n + \kappa n_{th} (n + 1) p(n) \quad (183)$$

---

Interest of the Monte Carlo method:

- For each trajectory computes only a state vector with $N_S$ dimensions i.e. $N_S$ coupled differential equations, instead of $N_S^2$ equations for the full density operator.
- Needs a statistical sample of trajectories. A few hundreds is enough to get a qualitative solution. Method more efficient than the direct integration when $N_S$ is larger than a few hundreds.
- Gives a physical picture of the relaxation process (see below).

An extremely useful method, with thousands of applications.
Field relaxation

Thermal equilibrium

Detailed balance argument

\[ \kappa(1 + n_{\text{th}})np(n) = \kappa n_{\text{th}} np(n - 1) \]  \tag{184}

leading to:

\[ \frac{\rho(n)}{\rho(n-1)} = \frac{n_{\text{th}}}{1 + n_{\text{th}}} = e^{-\hbar \omega / k_B T} \]  \tag{185}

The expected Maxwell equilibrium

At \( T = 0 \), relaxation of a Fock state

- Jump: removal of a photon
- No jump: non hermitian Hamiltonian

\[ H_e = -i\hbar J = -i\hbar c a^\dagger a / 2 \]  \tag{186}

Leaves photon number states invariant

Field relaxation

Fock states

Monte Carlo trajectory

- Jump: no evolution since \( |\alpha\rangle \) is an eigenstate of \( a \)
- No jumps: evolution with non hermitian hamiltonian, equivalent to a complex mode frequency

\[ |\beta\rangle \rightarrow \beta e^{-\kappa \tau / 2} \]  \tag{187}

A coherent state remains coherent, with an exponentially damped amplitude.
No change of the photon number in a quantum jump? A Bayesian argument. $p(n|c)$ photon number distribution before the jump knowing that a jump occurs (‘click’ in the environment.) With

$$p(n, c) = p(c|n)p(n) = p(n|c)p_c$$  \hspace{1cm} (188)$$

$$p(n|c) = p(n)p(c|n) = n! \frac{p(n)}{\pi^c} = e^{-\pi} \frac{\pi^{n-1}}{(n-1)!} = p(n-1)$$  \hspace{1cm} (189)$$

A translated Poisson distribution with $\pi + 1$ photons on the average. After jump photon number unchanged. Explains why the photon number distribution is invariant in a jump. Specific property of coherent states.
Outline

1. Spontaneous emission in free space
2. Photodetection
3. The dressed atom model
Spontaneous emission in free space

Coupling a quantized atom to the continuum of quantized modes in free space leads to a finite lifetime for an excited level $|e\rangle$ (only the ground state is stable).

Generally many downward transitions from $|e\rangle$. All rates of these transitions add up independently (no quantum interference: final states are different). One can thus compute the rate of a single transition between $|e\rangle$ and a lower non-degenerate state $|g\rangle$.

- Write the atom-field Hamiltonian
- Solve the Schrödinger equation (Wigner Weisskopf approach)
- Compute emission rate and (possible) energy shifts

Wigner Weisskopf approach

Atom-field state at time $t$:

$$|\Psi(t)\rangle = c_0(t) |e,0\rangle + \sum_{\ell} c_{\ell}(t) |g,1_{\ell}\rangle$$  \hspace{1cm} (4)

Schrödinger equation:

$$i\hbar \frac{dc_0}{dt} = \hbar \omega_e c_0 + \sum_{\ell} V_{\ell} c_{\ell}$$  \hspace{1cm} (5)

$$i\hbar \frac{dc_{\ell}}{dt} = \hbar \omega_{1_{\ell}} c_{\ell} + V_{\ell}^* c_0$$  \hspace{1cm} (6)

$$V_{\ell} = -\langle e,0|\mathbf{D}\cdot\mathbf{E}|g,1_{\ell}\rangle$$  \hspace{1cm} (7)

Interaction representation:

$$b_{\ell} = c_{\ell} e^{i\omega_{1_{\ell}} t}$$

$$i\hbar \frac{db_{\ell}}{dt} = e^{i\omega_{1_{\ell}} t} V_{\ell}^* c_0$$  \hspace{1cm} (8)

Setting

$$c_0 = e^{-i\omega_e t} \alpha_0(t)$$  \hspace{1cm} (11)

We get

$$\frac{d\alpha_0}{dt} = -\sum_{\ell} \frac{|V_{\ell}|^2}{\hbar^2} e^{i\omega_{1_{\ell}} t} \int_0^t e^{i\omega_{1_{\ell}} (t'-t)} e^{-i\omega_e t'} \alpha_0 dt'$$  \hspace{1cm} (12)
Wigner-Weisskopf

Changing for the variable $\tau = t - t'$, we get

$$\frac{d\alpha_0}{dt} = - \int_0^t N(\tau)\alpha_0(t - \tau) \, d\tau$$

(13)

where the integral kernel $N$ is:

$$N(\tau) = \frac{1}{\hbar^2} \sum_{\ell} |V_\ell|^2 e^{i(\omega_\ell - \omega_0)\tau}$$

(14)

Thus:

$$\int_0^t N(\tau)\alpha(t - \tau) \, d\tau \approx \alpha_0(t) \int_0^\infty N(\tau) \, d\tau = \left( \frac{\Gamma}{2} + i\Delta \right) \alpha_0(t)$$

(17)

$$\frac{d\alpha_0}{dt} = - \left( \frac{\Gamma}{2} + i\Delta \right) \alpha_0$$

(18)

- $\Gamma$ spontaneous emission rate
- $\Delta$ level shift

Wigner-Weisskopf

Let us calculate $V_\ell = - \langle e, 0 | D \cdot E | g, 1_\ell \rangle$.

Without loss of generality $D = qZu_z$.

Using $E(0) = i \sum_{\ell} \sqrt{\frac{\hbar\omega_\ell}{2\epsilon_0}} \phi_\ell e + \text{h.c.}$ we get

$$|V_\ell|^2 = \frac{\hbar\omega_\ell}{2\epsilon_0 V} |d|^2 |u_z \cdot \phi_\ell|^2$$

(15)

with $d = \langle e | qZ | g \rangle$.

Hence,

$$N(\tau) = \left[ \sum_{\ell} |u_z \cdot \phi_\ell|^2 \frac{\hbar\omega_\ell}{2\epsilon_0 V} e^{-i\omega_\ell \tau} \right] e^{i\omega_\ell \tau}$$

(16)

In the square brackets, a sum of very many oscillations with a large span of frequencies. In a time of the order of $1/\omega_\ell$, $N$ practically vanishes.

Final solution

$$c_0(t) = e^{-\Gamma t/2} e^{-i\omega_0 t} e^{-i\Delta t}$$

(19)

$$c_\ell(t) = \frac{V_\ell}{\hbar \left( \Gamma/2 \right) - i(\omega_\ell - \omega_0 - \Delta)}$$

(20)

$$|c_\ell(\infty)|^2 = \frac{|V_\ell|^2}{\hbar^2 \left( \Gamma^2/4 \right) + (\omega_\ell - \omega_0 - \Delta)^2}$$

(21)

a lorentzian profile for the spontaneous emission line.
Wigner-Weisskopf

Spontaneous emission in free space

Explicit integration of the kernel:

\[
\left( \frac{\Gamma}{2} + i \Delta \right) = \frac{|d|^2}{\hbar^2} \sum \epsilon z (u_z \cdot \epsilon_z)^2 \frac{\hbar \omega_f}{2 \epsilon_0 V} \int_0^\infty e^{i(\omega_{eg} - \omega_f)t} \, dt \tag{22}
\]

Using

\[
\int_0^\infty e^{i\omega t} \, dt = \pi \delta(t) + i \mathcal{P} \frac{1}{\omega} \tag{23}
\]

we get for the real part:

\[
\Gamma = \frac{2\pi |d|^2}{\hbar^2} \sum \epsilon z (u_z \cdot \epsilon_z)^2 \frac{\hbar \omega_f}{2 \epsilon_0 V} \delta(\omega_{eg} - \omega_f) \tag{24}
\]

Wigner-Weisskopf

Plane-wave basis: counting the modes

\[ N_{\nu} \text{ the total number of modes } k < 2\pi \nu / c. \text{ Number of modes between } \nu \text{ and } \nu + d\nu: \rho_{\nu} \, d\nu = \frac{dN_{\nu}}{d\nu} \tag{26} \]

Counting the modes with a frequency lower than \( \nu \) amounts to counting twice (two polarizations) the number of points with integer coordinates in a sphere of radius \( 2\pi \nu / c \):

\[ N_{\nu} = 2 \frac{4\pi (2\pi \nu / c)^3}{8\pi^3} = \frac{8\pi \nu^3}{3c^3} \tag{27} \]

Hence

\[ \rho_{\nu} = \frac{8\pi}{c^3} \nu^2 \tag{28} \]
Wigner-Weisskopf

Spontaneous emission rate

Expand $u_z$ on the basis of $u_k$ (propagation direction) and two orthogonal linear polarizations $\epsilon_1$ and $\epsilon_2$:

$$(u_z \cdot \epsilon_1)^2 + (u_z \cdot \epsilon_2)^2 = 1 - (u_z \cdot u_k)^2 = 1 - \cos^2 \theta = \sin^2 \theta \quad (30)$$

Integration over solid angle:

$$\int \sum _{\epsilon_\ell} (u_z \cdot \epsilon_\ell)^2 d\Omega = \int _0 ^{2\pi} \int _0 ^{\pi} \sin ^3 \theta d\theta d\phi = \frac{8\pi}{3} \quad (31)$$

and finally

$$\Gamma = \frac{|d|^2 \omega _{eg}^3}{3\pi\omega _0 \hbar ^3} \quad (32)$$

Photodetection

Photodetector model

A simple single-system photodetector. A ground state $|g\rangle$ and a continuum of excited states $|e_i\rangle$. Transition to excited state is a click.

Detector Hamiltonian

$$H_d = \sum _i \hbar \omega _i |e_i\rangle \langle e_i| \quad (33)$$

Detector-field interaction $-D \cdot E$ with

$$D = \sum _i d_i (\epsilon_i |g\rangle \langle e_i| + \epsilon_i^* |e_i\rangle \langle g|) \quad (34)$$

Hence, within irrelevant factors

$$H_i = \sum _i \kappa _i |e_i\rangle \langle g| E^+ + \text{h.c.} \quad (35)$$

Photodetection model

Interaction representation $a_\ell \rightarrow a_\ell \exp (-i\omega _\ell t) E \rightarrow E(t)$,

$$|e_i\rangle \langle g| \rightarrow \exp (i\omega _\ell t) |e_i\rangle \langle g| \quad (36)$$

Initial condition

$$|\Psi (0)\rangle = |g\rangle \otimes |\Psi _f\rangle \quad (37)$$

State at time $t$

$$|\Psi (t)\rangle = |g\rangle \otimes |\Psi _f\rangle + \frac{1}{i\hbar} \int _0 ^t \widetilde{H}_i (t') |\Psi (t')\rangle \ dt' \quad (38)$$

First-order perturbative solution by replacing in the r.h.s. $|\Psi (t')\rangle$ by $|\Psi (0)\rangle = |g\rangle \otimes |\Psi _f\rangle$. 
Photodetector model

Noting that, in $\hat{H}$, $|g\rangle \langle e| E^- \gives 0$ on the initial state

$$|\Psi(t)\rangle = |g\rangle \otimes |\Psi_f\rangle + \frac{1}{\hbar} \sum_i k_i \left[ \int_0^t dt' e^{i\omega(t-t')} E^+(t') |\Psi_f\rangle \right] \otimes |e_i\rangle$$ (39)

Probability for having a count at time $t$

$$p_e = \sum_i |\langle e_i | \Psi \rangle|^2 = \sum_i \langle \Psi | e_i \rangle \langle e_i | \Psi \rangle$$ (40)

$$p_e = \frac{1}{\hbar^2} \sum_i k_i^2 \int_0^t dt' \int_0^{t'} dt'' e^{i\omega(t'-t'')} \langle \Psi_f | E^- (t'') E^+(t') |\Psi_f\rangle$$ (41)

The dressed atom model

A frequent situation in quantum optics: a two-level atom coupled to a single mode of the radiation field. Coherent coupling larger than dissipative process.
- An atom in an intense laser field
- Cavity quantum electrodynamics

An ideal situation
A two-level atom coupled to a single field mode. Or a spin 1/2 coupled to a harmonic oscillator. The simplest non-trivial quantum system.

Almost ideally implemented in Cavity Quantum Electrodynamics, circuit QED but also in ion traps. Much more on that soon.

Fruitful to treat atom and mode as a single quantum system: the Dressed atom (Cohen-Tannoudji)

Dressed atom

A two-level system

We consider the case of a single radiation mode at frequency $\omega_0$ resonant or nearly resonant on the transition between the two levels $|g\rangle$ (lower, possibly ground level) and $|e\rangle$ i.e.

$$\omega_0 \approx \omega_{eg}$$

All other levels can be neglected.
The dressed atom model
Free atom
Two states $|e\rangle$ and $|g\rangle$ or $|\pm\rangle$ and $|0\rangle$ and $|1\rangle$ in quantum information science. Equivalent to a spin-1/2 system.
Operator basis set: Pauli operators

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(46)

$$[\sigma_x, \sigma_y] = 2i\sigma_z$$

(47)

Spin lowering and raising operators

$$\sigma_+ = |\pm\rangle \langle +| = \frac{\sigma_x + i\sigma_y}{2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(48)

$$\sigma_- = |\mp\rangle \langle -| = \frac{\sigma_x - i\sigma_y}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

(49)

$$[\sigma_x, \sigma_\pm] = \pm 2\sigma_\mp$$

(50)

Eigenvectors

$$|\pm u\rangle = |\pm 0\rangle = \cos \frac{\theta}{2} |+\rangle + \sin \frac{\theta}{2} e^{i\phi} |-\rangle$$

(52)

$$|-u\rangle = |1\rangle = -\sin \frac{\theta}{2} e^{-i\phi} |+\rangle + \cos \frac{\theta}{2} |\mp\rangle$$

(53)

Span the whole Hilbert space for all values of $\theta$ and $\phi$.

Going from one state to another by a rotation on the Bloch sphere by an angle $\theta$ around the axis defined by $\mathbf{v}$

$$R_\mathbf{v}(\theta) = e^{-i(\theta/2)\mathbf{v} \cdot \sigma} = \cos \frac{\theta}{2} \mathbf{1} - i \sin \frac{\theta}{2} \mathbf{v} \cdot \sigma$$

(54)

e.g. angle $\theta$ around $\mathbf{u}_z$

$$R_z(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

(55)

with $R_z(\pi/2) |+\rangle = |+\rangle$ and $R_z(2\pi) = \mathbf{1}$.
Free atom

- Hamiltonian:
  \[ H_a = \frac{\hbar \omega_{eg}}{2} \sigma_z \]  
  (56)
  Generates a rotation of the Bloch vector at angular frequency \( \omega_{eg} \) around \( OZ \) (Larmor precession in the NMR context).
- Dipole operator:
  \[ D = d(\epsilon_a \sigma_+ + \epsilon_a^* \sigma_-) \]  
  (57)
  where \( \epsilon_a \) describes the polarization of the atomic transition.

Atomic relaxation

Spontaneous emission can be treated as a perturbation (weak effect compared to atom-field interaction). Described in terms of quantum jumps and Lindblad equations. We assume a zero-temperature environment (valid for optical transitions).

A single jump operator (describing photon emission in a downwards transition)

\[ L = \sqrt{\Gamma} \sigma_- \]  
(58)

with \( \Gamma = 1/T_1 \) ('longitudinal relaxation time'). Lindblad equation

\[ \frac{d\rho}{dt} = \Gamma \left( \sigma_- \rho \sigma_+ - \frac{1}{2} \sigma_+ \sigma_- \rho - \frac{1}{2} \rho \sigma_+ \sigma_- \right) \]  
(59)

Atomic relaxation

Case of an initial superposition state \( |\Psi_0\rangle = (1/\sqrt{2})(|e\rangle + |g\rangle) \). Analysis in terms of the Monte Carlo trajectories.

- No jump evolution. With \( |\Psi(t)\rangle = c_e |e\rangle + c_g |g\rangle \), we use the effective Hamiltonian

\[ H = -i\hbar J = -i\hbar \sigma_+ \sigma_- = -i\hbar \Gamma |e\rangle \langle e| \]  
(63)

\[ \frac{d c_e}{dt} = -i\hbar \sigma_- c_e \quad c_e(t) = c_e(0) e^{-\Gamma t/2} \]  
(64)

\[ |\Psi(t)\rangle = \frac{1}{|c_e(0)|^2 e^{-\Gamma t/2} + |c_g(0)|^2} \left( c_e(0) e^{-\Gamma t/2} |e\rangle + c_g(0) |g\rangle \right) \]  
(65)

A negative detection (no photon emitted) changes the system's state.

- Jump: state becomes \( |g\rangle \). No further evolution.
The dressed atom model

Field normalization

A slightly different choice of normalization for the spatial mode function. In free space: the volume is that of the fictitious quantization box. For an actual mode, or a cavity mode with a known geometry, less ambiguous to set

\[ f(r) = 1 \]  

(66)

at the field maximum. This defines the mode volume as

\[ V = \int |f(r)|^2 d^3r \]

(67)

Since we will soon refer explicitly to a cavity mode with angular frequency \( \omega_c \), use an index \( c \) for field operators.

Atom-field interaction

For the sake of simplicity, we assume that the atom is sitting at the mode center (the field maximum) corresponding to \( f(r) = 1 \). Then,

\[ H_{ac} = -i\hbar \frac{\Omega_0}{2} [a \sigma_+ - a^\dagger \sigma_-] \]

(71)

with

\[ \Omega_0 = \frac{2 d E_0 (\epsilon_a \sigma_- + \epsilon_b^* \sigma_+)}{\hbar} \]

(72)

\( \Omega_0 \) is called the “vacuum Rabi frequency” and we will (without loss of generality) assume it to be real. 

\( H \) is the Jaynes-Cummings Hamiltonian (1963).

Let us first examine the uncoupled quantum states.

Uncoupled states

The eigenstates of \( H_a + H_c' \) are the uncoupled states \( |e, n\rangle \) and \( |g, n\rangle \). Assuming \( \Delta_c = \omega_{eg} - \omega_c \ll \omega_{eg} \), all levels \( |e, n\rangle \) and \( |g, n + 1\rangle \) are nearly degenerate (energy separation \( \hbar \Delta_c \)). The ground state \( |g, 0\rangle \) is an isolated one, obviously impervious to the atom-mode interaction.
Uncoupled states

Dressed states

The dressed states, eigenstates of the full Hamiltonian have the energies

$$E_n^\pm = \pm \frac{\hbar}{2} \sqrt{\Delta_c^2 + \Omega_n^2}$$  \hspace{1cm} (75)

They are

$$|\pm, n\rangle = \cos \theta_n^\pm |e, n\rangle + i \sin \theta_n^\pm |g, n+1\rangle$$  \hspace{1cm} (76)

with

$$\tan \theta_n^\pm = \pm \frac{\sqrt{\Delta_c^2 + \Omega_n^2}}{\Omega_n} \mp \Delta_c$$  \hspace{1cm} (77)

The expressions are generally complex. Simple cases at exact resonance ($\Delta_c = 0$) and in the dispersive, nonresonant regime ($\Delta_c \gg \Omega_n$)

Dressed states

Diagonalizing the Hamiltonian

$H_{ac}$ only couples the nearly degenerate states $|e, n\rangle$ and $|g, n + 1\rangle$. The diagonalization of the total Hamiltonian can be performed as separate diagonalizations of $2 \times 2$ Hamiltonians.

Use an interaction representation setting the energy origin at the center of the manifold. Final matrix form of the total Hamiltonian $H$ in the $\{|e, n\rangle, |g, n + 1\rangle\}$ basis

$$\hat{H} = \frac{\hbar}{2} \begin{pmatrix} \Delta_c & -i \Omega_n \\ i \Omega_n & -\Delta_c \end{pmatrix}$$  \hspace{1cm} (73)

where

$$\Omega_n = \Omega_0 \sqrt{n + 1}$$  \hspace{1cm} (74)

is the $n$-photon Rabi frequency

Dressed states

Resonant regime

At resonance

$$|\pm, n\rangle = \frac{1}{\sqrt{2}} \left[ |e, n\rangle \pm i |g, n + 1\rangle \right]$$  \hspace{1cm} (78)

Dressed states are equal-weight superpositions of the uncoupled states. The splitting of the dressed manifold is $\hbar \Omega_n$.

System initially prepared in $|e, n\rangle$: Rabi oscillation between the two uncoupled states at $\Omega_n$. 
The dressed atom model

Dressed states

Resonant regime

\[ |e, 2 \rangle \quad \Omega \sqrt{3} \quad |+, 2 \rangle \]

\[ |e, 1 \rangle \quad \Omega \sqrt{2} \quad |+, 1 \rangle \]

\[ |e, 0 \rangle \quad \Omega \quad |+, 0 \rangle \]

\[ |g, 0 \rangle \quad "\text{vacuum Rabi splitting}" \]

Classical Rabi oscillation

Case of a large coherent field \( \pi \gg 1 \). We can neglect the variation of \( \Omega_n \) over the range of populated \( n \) values. The dressed states multiplicities have then a constant splitting \( \Omega_r = \Omega_0 \sqrt{n} \).

The field state is not affected by atomic emission/absorption and we recover the classical Rabi oscillation at \( \Omega_r \) as a quantum beat between dressed states.

Autler-Townes splitting

Probe the dressed level structure by inducing the \( |h \rangle \rightarrow |e \rangle \) transition (\( h \) is a level uncoupled to the single mode).

A doublet of lines separated by \( \Omega_r \).

Mollow triplet

Add atomic relaxation (spontaneous emission). Both dressed states have an \( e \) character and can thus both decay towards dressed states in the immediately lower manifold (the uncoupled states decay is from \( |e, n \rangle \) to \( |g, n \rangle \)).

The emission spectrum is a triplet of lines (Mollow Phys. Rev. 188 1969 (1969))
The dressed atom model

Dressed states

Position of the dressed states as a function of the detuning.

\[ \Delta_c / \Omega_n(r) \]

\[ |e,n\rangle \]

\[ |+n,n\rangle \]

\[ |-,n\rangle \]

\[ |g,n+1\rangle \]

\[ E_n^\pm = \pm \hbar \left( \frac{\Delta_c}{2} + \frac{\Omega_c^2}{4\Delta_c} \right) \] (79)

In other words, \(|e,n\rangle\) is shifted by \(\hbar s(n+1)\) and \(|g,n+1\rangle\) is shifted by \(-\hbar sn\), where

\[ s = \frac{\Omega_c^2}{4\Delta_c} \] (80)

Dressed states

Dispersive regime

Interpretation: light and index shifts.

- Lamb and Light shifts: the atomic frequency is modified by

\[ \delta \omega_{eg} = s(2n + 1) \]

- Atomic index of refraction:

\[ \delta \omega_c = s \]

and

\[ \delta \omega_c = -s \]
Let us now discuss all that in more details in the Cavity Quantum Electrodynamics context.
Field quantization: cavity quantum electrodynamics

J.M. Raimond
Sorbonne Université
LKB, Collège de France, ENS, CNRS, SU

Outline of this lecture
• Introduction
• Tools of cavity QED
• Resonant interaction
• Dispersive interaction
• Perspectives

Cavity Quantum Electrodynamics
• A spin and a spring
  – Realizes the simplest matter-field system: a single atom coherently coupled to a few photons in a single mode of the radiation field.
  – Direct illustrations of quantum postulates

A history of CQED: the origin
• Purcell 1946
  – spontaneous emission rate modification for a spin in a resonant circuit
  – Definition of the ‘Purcell factor’
  – Brief but seminal
• Kleppner 81
  – Inhibition of spontaneous emission

B:10. Spontaneous Emission Probabilities at Radio Frequencies. E. M. Purcell, Harvard University.—For nuclear magnetic moment transitions at radio frequencies the probability of spontaneous emission, computed from

\[ J_s = \frac{n\nu^2}{6\pi^2\hbar^3\nu_{\text{en}}} \text{sec}^{-1} \]

is so small that this process is not effective in bringing a spin system into thermal equilibrium with its surroundings. At 300°C, for \( s = 10^6 \text{ sec}^{-1} \), \( \omega = 1 \text{ nuclear magneton} \), the corresponding relaxation time would be \( 5 \times 10^6 \text{ seconds} \). However, for a system coupled to a resonant electric circuit, the factor \( \nu^2/\hbar^3 \) no longer gives correctly the number of radiation oscillators per unit volume, in small frequency range, there being now an oscillator in the frequency range \( \nu_{\text{en}} \) associated with the circuit. The spontaneous emission probability is thereby increased and the relaxation time reduced, by a factor \( 1 + 3\nu_{\text{en}}/\nu^2 \) where \( V \) is the volume of the resonator. If the resonant characteristic of the circuit so that \( V = \nu_{\text{en}} \), and if it is the slice-depth at frequency \( \nu \), \( \nu = 3\nu^2/\nu_{\text{en}} \). For a non-resonant circuit \( J_s = \nu^2/\hbar^3 \) and for \( s = 10^6 \text{ sec}^{-1} \). If small metallic particles, of diameter \( 10^{-6} \text{ cm} \) are mixed with a nuclear magnetic medium at room temperature, spontaneous emission should establish thermal equilibrium in a time of the order of minutes, for \( s = 10^6 \text{ sec}^{-1} \).
The two regimes of cavity QED

- **Weak coupling regime**
  - Atom-field coupling small compared to dissipation
    - No qualitative modifications of the atomic radiative properties
    - Modification of the spontaneous emission rate
    - Modification of the atomic energies

- **Strong coupling regime**
  - Atom-cavity interaction overwhelms dissipative processes
    - The simplest matter-field coupling situation
      - Radical modification of the atomic radiative properties
      - Creates and manipulates atom/field entangled state

First experiment on weak coupling

- **Spontaneous emission enhancement**
  - Superconducting FP cavity
    - $Q \alpha 10^6$
    - 340 GHz transition
  - Acceleration x 530
  - First experimental evidence of Purcell effect

- **Spontaneous emission inhibition**
  - Gabrielse and Dehmelt (85)
  - Hulet, Hilfer and Kleppner (85)
  - Spontaneous emission can be altered at will by imposing limiting conditions to the field

First experiment on strong coupling: the micromaser

- H. Walther and D. Meschede, 85
  - Cumulative emissions in the cavity in the strong coupling regime
    - A maser with less than one atom at a time in the cavity
    - A new type of quantum oscillator. Role of quantum fluctuations
  - Strong coupling regime
    - Single-Atom-cavity coupling overwhelms dissipation

The four time scales of CQED

- **Atomic levels lifetime**
  \[ T_a = \frac{1}{\Gamma} \]

- **Cavity lifetime**
  \[ T_c = \frac{1}{\kappa} \]

- **Atom-cavity coupling**
  \[ \Omega_{0} = 2g = \frac{1}{T_{res}} \]

- **Atom-cavity interaction time**
  \[ T_{int} \]

- **Strong coupling conditions**
  \[ T_{int} \Omega_{0} = 1; \quad T_{res}, T_{int} \ll T_a, T_c \]
The four flavours of modern CQED

- **Optical CQED**
  - Ordinary atomic transitions and high finesse FP cavities
  
  \[ g \approx 50 \text{ MHz}; \; \kappa \approx 100 \text{ kHz}; \; \Gamma \approx 10 \text{ MHz}; \; T_{int} \approx 1\text{s} \]

- **Solid-state CQED**
  - Quantum dots coupled to bragg mirrors/PBG
  
  \[ g \approx 10 \text{ GHz}; \; \kappa \approx 1 \text{ GHz}; \; \Gamma \approx 1\text{GHz}; \; T_{int} = \infty \]

- **Circuit QED**
  - Solid-state qubits and stripline cavities
  
  \[ g \approx 100 \text{MHz}; \; \kappa \approx 1 \text{ GHz}; \; \Gamma \approx 10 \text{ Hz}; \; T_{int} = \infty \]

- **Microwave CQED**
  - (Circular) Rydberg atoms and superconducting cavities
  
  \[ g \approx 10 \text{ kHz}; \; \kappa \approx 1 \text{ Hz}; \; \Gamma \approx 30 \text{ Hz}; \; T_{int} \approx 100 \mu\text{s} \]

Outline of this lecture

- Introduction
- Tools of cavity QED
- Resonant interaction
- Dispersive interaction
- Perspectives

These lectures

- Focus on microwave QED
  - Paradigmatic example of CQED
  - Some (hopefully) interesting experiments

Microwave CQED experiments at ENS (1983–…)

- Two ideal tools
  - Circular Rydberg atoms
    - Ideal two-level atoms
      - Long lifetime (30ms)
      - Microwave two-level transition
      - Stark tuning
      - Huge dipole matrix element
      - Selective and sensitive detection
        - Field ionization
      - Poissonian atom number in a laser excited sample
        - No deterministic one atom preparation
  - Superconducting microwave Fabry Perot cavity
    - A nearly ideal photon box
      - \( T_i = 0.13 \text{s at } 0.8 \text{ K} \):
        - a macroscopic time interval
      - \( Q=4.2 \times 10^9, \; F=4.6 \times 10^9 \)
Mirror technology

- Copper substrates
  - diamond machining
  - shape accuracy 300 nm ptv
  - rugosity 10 nm
  - Toroidal surface ➔ single mode

- 12 µm Niobium layer
  - Cathode plasma sputtering
  - CEA, Saclay
  - [E. Jacques, B. Visentin, P. Bosland]
  - S. Kuhr et al, APL, 90, 164101

Experimental set-up

- Circular Rydberg atoms
- Microwave cavity
- Ramsey interferometer
- Circular state preparation
- Field ionization detector
- Ramsey zones
- 5 cm atomic detectors
- Off resonance
- Screening B + thermal fields
- Two cavities
- 40 kg of copper at 0.8 K
A CQED experiment with slow atoms

- Thermal beam
  - Atomic velocity 200-500 m/s, interaction time limited to 80 µs
  - A strong limitation for some experiments
- A cavity QED experiment in an atomic fountain configuration
  - 10 m/s velocity, very long interaction times

The tools of circuit QED

- Atoms
  - Superconducting circuits based on Josephson junctions
  - Most popular example: the transmon
    - A weakly nonlinear harmonic oscillator
    - Long relaxation times up to tens of µs

- Cavities
  - Superconducting stripline resonators on a chip
  - 3D Pill-box superconducting cavities

A complete set-up

- A thriving field worldwide
  - Schoelkopf, Devoret (Yale); Martinis (Santa Barbara); Wallraff (ETH) Esteve (CEA); Nakamura (NTT)…
Circuit and Cavity QED

- Two totally different experimental approaches
  - Tools of AMO physics or solid-state
  - Both in the microwave (5-50GHz domain)

- Very different orders of magnitude
  - Much higher atom-field coupling in circuit QED but much faster dissipative process
  - Much longer atom-cavity interaction time in circuit QED

- But, *mutatis mutandis*, similar capabilities and achievements.
  - All the discussions in these lecture apply to the cavity QED context but can be immediately transposed to circuit QED
  - And, in a large part to trapped ions

Cavity mode volume

- A Fabry perot resonator with a Gaussian standing-wave mode
  - Directly described by the formalism of the previous lectures
    - Normalization: $f$ is 1 at the field maximum

\[ V = \int |f(r)|^2 \, d^3r = \frac{\pi w_0^2 L}{4} \]

\[ V = 0.7 \text{ cm}^3, \quad E_0 = 1.5 \text{ V/m} \]

Taking into account atomic motion: effective interaction time

- Real atoms cross gaussian mode: vacuum Rabi frequency is a function of time

\[ \Omega_0 f(vt) \quad f(\nu t) = e^{-\nu^2 t^2 / w_0^2} \]

- Simple expressions only in resonant and dispersive cases
  - Resonant case
    - Interaction from $t_i$ to $t_f$
    - For a full cavity transit
      \[ T = \int f(\nu t) \, dt = \sqrt{\frac{\pi w_0}{\nu}} \]
    - Replace everywhere $\Omega_0(n)(\mathbf{r})$ by $\Omega_0 \sqrt{n+1}$ and use effective interaction time

- Dispersive case
  - Define effective Hamiltonian, proportional to $|f|^2$
    \[ H_{\text{eff}}(\mathbf{r}) = \hbar \Omega_0 |f(\mathbf{r})|^2 \left( \sigma_z (a^\dagger a + \frac{1}{2}) + \frac{1}{2} \right) \]
  - Full cavity transit
    \[ T^d = \exp \left[ -(i/\hbar) H_{\text{eff}}(0) t_f^d \right] \]
  - Use the effective interaction time and the shifts at cavity centre
    - Note that resonant and non-resonant effective interaction times are not equal
Outline of this lecture

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  - Vacuum Rabi oscillation
  - Rabi in a mesoscopic field
- Dispersive interaction
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Vacuum Rabi oscillations

- The simplest situation: initial state $|e, 0\rangle$

- Periodic exchange of one oscillation quantum between the atom and the cavity at the 'vacuum Rabi frequency' $\Omega_0$.
- Time counterpart of the 'vacuum Rabi splitting'
- Observable only in the strong coupling regime, when oscillation frequency is much higher than the dissipation rates

An early experiment

Quantum Rabi oscillations: state transformations

- $\pi/2$ spontaneous emission pulse
- Entanglement creation
- Atom-cavity EPR pair

Initial state $|e, 0\rangle$
Quantum Rabi oscillations: state transformations

Brune et al, PRL 76, 1800 (96)

Three "stitches" to "knit" quantum entanglement

Combine elementary transformations to create entangled states
an early exploration of quantum logics

- State copy with a π pulse
  - Quantum memory: PRL 79, 769 (97)
- Creation of entanglement with a π/2 pulse
  - EPR atomic pairs: PRL 79, 1 (97)
- Quantum phase gate based on a 2π pulse
  - Quantum gate: PRL 83, 5166 (99)
  - Absorption-free detection of a single photon: Nature 400, 239 (99)
- Entanglement of three systems (six operations on four qubits)
  - GHZ Triplets: Science 288, 2024 (00)
- Entanglement of two radiation field modes
  - Phys. Rev. A 64, 050301 (2001)
- Direct entanglement of two atoms in a cavity-assisted collision

A vacuum Rabi experiment with slow atoms

F. Assenmat et al arXiv:1905.05247
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Rabi oscillation in a mesoscopic coherent field

• Intermediate regime of a few tens of photons.
  – A simple theoretical problem
    \[ |\Psi(t)\rangle = \sum_n c_n |n\rangle \]
    \[ = \sum_n c_n \cos \frac{\Omega_0 \sqrt{n + 1} t}{2} |n\rangle + c_n \sin \frac{\Omega_0 \sqrt{n + 1} t}{2} |g, n + 1\rangle \]
  – A surprisingly complex behavior
    • Computed for 15 photons

Collapse and revival

• Collapse:
  – dispersion of field amplitudes due to dispersion of photon number
    \[ t_c \approx \frac{\pi}{\Omega_0} \]
• Revival:
  – rephasing of amplitudes at a finite time such that oscillations corresponding to \( n \) and \( n+1 \) come back in phase
    \[ t_r \approx \frac{4\pi}{\Omega_0} \sqrt{n} \]
  – Revival is a genuinely quantum effect
  – Get a better understanding

Observation with slow atoms

• Clear revivals in a 13 photon field!
  – Excellent agreement with a numerical simulation taking into account a few experimental imperfections (detection errors, residual thermal field...)

F. Assemat et al. PRL in print arXiv:1905.05247
Field amplitude quantization

Revival signal Fourier Transform reveals discrete frequencies
- Direct evidence of field quantization
- Direct measurement of photon number distribution

An insightful quasi-exact solution

\[ |\Psi(t)\rangle = \frac{1}{\sqrt{2}} \left[ |\Psi_+\rangle(t) |\Psi_-\rangle(t) + |\Psi_-\rangle(t) |\Psi_+\rangle(t) \right] \]

\[ |\Psi_z\rangle = \frac{1}{\sqrt{2}} e^{i\alpha_0 \sqrt{n}/2} \left[ e^{i\phi} |e\rangle \mp i |g\rangle \right] \quad \Phi = \frac{\Omega J}{4\sqrt{n}} \]

- Atomic states slowly (n times slower than Rabi oscillation) rotating in the equatorial plane of the Bloch sphere
- A slowly rotating field state in the Fresnel plane

Graphical representation of the joint atom-field evolution in a plane
- t=0:
  - both field states coincide with original coherent state
  - Atomic states are the classical eigenstates

Link with Rabi oscillation

Rabi oscillation: quantum interference between |\Psi_+\rangle and |\Psi_-\rangle
- Contrast vanishes when \langle \Psi_+ | \Psi_- \rangle = 0:
  - A direct link between Rabi collapse and complementarity

Atom-field states evolution

|\Psi_+\rangle \quad |\Psi_-\rangle \quad |\psi_0\rangle \quad \psi_0 = \frac{1}{\sqrt{2}} \left[ |\psi_+\rangle |\psi_-\rangle + |\psi_-\rangle |\psi_+\rangle \right] \quad \langle \psi_0 | \psi_0 \rangle = 0

At most times: \langle \Psi_+ | \Psi_- \rangle = 0 an atom-field entangled state

In spite of large photon number: considerable reaction of the atom on the field
Generation of Schrödinger cat states

• At half revival, the atom is disentangled
  – Field left in a superposition of two coherent amplitudes with opposite phases
    $$|\psi_{\text{cat}}\rangle = e^{i\pi/\hbar} |\beta\rangle - |\beta\rangle$$
  – A Schrödinger cat states of the field (and a large one)
  – Resonant interaction leads to the fastest preparation of such cat states

• Check the cat:
  – At half revival, reset the atom in state $e$ and record its Rabi oscillation in the cat (with a small deterministic translation in phase space)

Ideal quantum measurement of the photon number

• Quantum discontinuity
  – Not all measurement results allowed
    • eg: number of photons is an integer

• Quantum indeterminacy
  – Predict only the probabilities of possible outcomes
  – Result of a measurement intrinsically random
    • God is playing dice
    • e.g. Poisson photon number distribution in a classical laser pulse

• Repeatability
  – Repetition of ideal measurements of a constant of motion always give the same result
    • Projection postulate.
    • Repeated measurements of the photon number always give the same result

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• Tools of cavity QED
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• Dispersive interaction
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  – Fock states
  – Complementarity and Schrödinger cats
  – Quantum feedback
• Perspectives
Ideal and real quantum measurements

- Most quantum measurements are far from ideal
  - e.g. Photodetection (counting photons)
    - measurement of light field energy
      - quantized result: number of photons
      - statistical: photon number statistics
      - repeatable?
  - Photodetectors absorb incoming photons
    - A second detection always gives zero: impossible to ‘see’ the same photon twice
    - The field state is demolished by the detection
  - This demolition is not a requirement of quantum mechanics

Ideal photon number counting

- Most quantum measurements are far from being projective
  - Light detection:
    - Photons are destroyed when detected
  - Quantum non-demolition measurements (Braginsky, 70s)
    - A transparent photocounter
      - ‘see’ the same photon twice
      - Should allow observation of the quantum jumps of light
  - Realized in the optical domain (Grangier et al, Nature, 396, 537)
    - no single photon resolution
    - weak non-linearity
    - propagating fields:
      - repetition difficult

Yet another gedankenexperiment

- A clock whose ticking rate is determined by the number of photons in a box
- The final clock hand’s position directly measures the photon number
  - Box: a superconducting millimetre wave cavity
  - Clock: a single circular Rydberg atom

Dispersive atom-field interaction

- Atomic frequency shift inside the cavity
  - Light and Lamb shifts:
    - An atomic clock ticking rate modification
    - A phase shift of the atomic coherence
  - Adiabatic coupling in and out of the atom-cavity interaction
    - negligible spurious absorption rate ($<10^{-4}$ for $\delta \sim \Omega$)
A pictorial representation of the interaction

- Evolution of the atomic state on the Bloch sphere.
  - π/4 phase shift per photon

- In general non-orthogonal final atomic states correspond to different photon numbers: A single atom does not tell all the story
- A simple case: π phase shift per photon and 0/1 photon (a ‘qubit’ situation)

The single photon case

- A zero or one-photon field
  - π phase shift per photon

- Two orthogonal final atomic states
  - in principle, a single atomic detection unambiguously tells the photon number.

Birth, life and death of a photon

T=0.8 K nth=0.05

1 sequence:

Excellent agreement with the quantum predictions (no adjustable parameter)

- In general non-orthogonal final atomic states correspond to different photon numbers: A single atom does not tell all the story.
Photon counting by information accumulation

- One atom exits cavity with a spin direction correlated to $n$
- QND interaction: $N$ atoms exit cavity with the same spin direction correlated to $n$
  - Entanglement of the photon number with a mesoscopic atomic sample

- Split atomic sample in two parts
  - On $N/2$ atoms, measure $S_x$
  - On $N/2$ atoms, measure $S_y$
  - Estimate spin direction with uncertainty $d_x = \frac{1}{\sqrt{N}}$

- A Bayesian inference process
  - Photon number distribution
    - Relaxation and measurement operators diagonal in the Fock states basis
      - Updated according to atomic detection results
      - Detection probabilities:
        $\pi_j(\phi_j|n) = \frac{P_j(\phi_j | n)}{\pi_j(\phi_j | p)} P_{j-1}(n)$
        $\pi_x(\phi_x|n) = 1 - \pi_x(\phi_x|n) = \frac{1}{2} (1 + \cos[\phi_x + \phi_0(n + 1/2)])$
      - Updated according to cavity relaxation
        $\frac{dP^f(n,t)}{dt} = \sum_{m} K_{nm} P^f(m,t)$
    - A Bayesian inference of $P(n)$ by photon decimation, proceeding forward in time
      - About $8^2$ atoms required to count from 0 to 7

"Forward" estimation of the photon number at time $t$

- Density operator $\rho$ including all available information from 0 to $t$
  - Updated according to each atomic detection result
    $\rho_{p-1}^f \rightarrow \rho_p^f = \frac{M_j \rho_{p-1}^f M_j^\dagger}{\pi_j(\phi_j | \rho_{p-1}^f)}$
  - Measurement operators
    $M_j = \sin \left[ \frac{\phi_j + \phi_0(N + 1/2)}{2} \right]$  $M_j = \cos \left[ \frac{\phi_j + \phi_0(N + 1/2)}{2} \right]$  $\tau_j(\phi_j | \rho) = \text{Tr} \left( M_j \rho M_j^\dagger \right)$
  - Updated according to cavity relaxation between detections
    - Liouvillian evolution

Single atom detection

Each detection brings partial information on the photon number

To speed-up convergence, the measurement phase $\phi$ is randomly chosen for each atom among the four values corresponding to atomic states
Wave-function collapse in real time

- Evolution of $P(n)$ while detecting 110 atoms in a single sequence
- Initial coherent field with 3.7 photons
- Initial inferred distribution flat (no information) but final result independent of initial choice
- Progressive collapse of the field state vector during information acquisition

Photon number statistics

Excellent agreement with the expected Poisson distribution
A vivid illustration of quantum measurement postulates

Cascade down the Fock states ladder

- Repeated measurements confirm the $n=7$ result
- Quantum jumps corresponding to field damping
- Quantum state projection onto $n=7$

A single quantum trajectory with a large initial field

- Forward estimation at time $t$
- Evident problems
  - Initial ambiguity in the photon number due to the periodicity of the measurement operators
  - Absurd photon number jumps (from 0 to 7)
  - Noise due to statistical fluctuations of atomic detections
- Improvement by taking into account measurements to come after $t$
The Past Quantum State approach

- A posteriori estimation of the photon number at $t$ based on all available information, gathered from 0 to $t$ AND from $t$ to $T$
  - From the journalist’s to the historian’s perspective
- A quantum formalism (S. Gammelmak et al. PRL 111, 160401)
  - The Past quantum state

  - Best estimate for the results of a quantum measurement at $t$ based on the density matrix $\rho$ computed forward in time AND on an effect matrix $E$ computed backwards in time

Forward-backward estimation

- For diagonal measurement/relaxation operators
  $$P^{F}(n,t) = \frac{P^{F}(n,t)P^{b}(n,t)}{\sum_{m} P^{F}(m,t)P^{b}(m,t)}$$
  - PQS reduces to the forward/backward smoothing algorithm, which can be safely used in this quantum context
  - $P(n)$ is the product of two photon number distributions computed forward and backward in time.

- Backwards estimation
  - Flat distribution at $T$
  - Same measurement operators
  - ‘Inverse’ relaxation (annihilation and creation operators exchanged)
    - Exponential growth of the photon number

PQS estimation

- Ambiguities lifted
  - Measurement of photon number beyond the intrinsic periodicity of atomic signal
- Considerable noise reduction
  - All estimations take into account ALL available information

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Quantum Zeno effect

- A watched kettle never boils
  - coherent evolution of a system and frequently repeated quantum measurements
    - a quantum jumps evolution between eigenstates of the measured quantity
    - an evolution much slower than without measurements
    - no evolution at all in the limit of zero delay between measurements
  - No Zeno effect for incoherent relaxation processes

Quantum Zeno effect

- Coherent evolution
  - Probability for leaving $|0\rangle$ quadratic in $t/n$
  - Efficient inhibition of coherent evolution

- Incoherent evolution (relaxation)
  - Probability for leaving $|0\rangle$ in the first step $\Gamma t/n$ (exponential decay)
  - Final probability for staying in $|0\rangle$ (assume $\Gamma t << 1$):
  
  $$ \left(1 - \frac{\Gamma t}{n}\right)^n \approx 1 - \frac{\Gamma t}{n} $$

  - Same decay without measurements
  - Zeno effect does not affect relaxation processes
  - Unless measurements frequently repeated on the scale of the environment’s correlation time

Quantum Zeno effect

- A simple description of the Zeno effect
  - A quantum system initially in $|0\rangle$ evolves under the action of the Hamiltonian $V$ during time $t$.
  - During this time, $n$ measurements of an observable $O$ with the non-degenerate eigenstate $|0\rangle$ are performed, at times $t/n, 2t/n, ...$
  - At $t/n$ the probability for finding $|0\rangle$ is

  $$ \prod_{j=1}^{n} \left[ 1 - \frac{\Delta V}{\hbar} \frac{t}{n} \right] \approx 1 - \frac{\Delta V t^2}{\hbar^2 n} $$

  - A quadratic function of the time interval $t/n$

  - Final probability for finding $|0\rangle$:

  $$ \prod_{j=1}^{n} \left[ 1 - \frac{\Delta V}{\hbar} \frac{t}{n} \right] \approx 1 - \frac{\Delta V t^2}{\hbar^2 n} $$

  - 1 if the time interval between measurements is close to zero.
  - Efficient inhibition of coherent evolution

Quantum Zeno effect

- Coherent evolution: injection of a coherent field by a classical source

  - Repeated injection of phase coherent pulses: an amplitude varying linearly with the number of injections (photon number varies quadratically).

  Principle of the experiment: perform QND measurements of photon number between two pulses
Growth of a coherent field

Photons injected per pulse:

\[ n_{\text{i}} = |\alpha_{\text{i}}|^2 = 0.00223 \pm 0.00012 \]

Random walk in phase-space

\[ A = N \cdot |\alpha_{\text{i}}|^2 \]

\[ n = A^2 = N^2 |\alpha_{\text{i}}|^2 \]

Quadratic growth

Linear growth

Residual field growth

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Fock state preparation

• Ideal projective measurement
  – After measurement the field is in a photon number state
  – Prepare all Fock states from 0 to 7
    • Not an easy task in quantum optics

• Check the produced state?
  – Full measurement of the cavity quantum state
    • Also based on the QND interaction
    • Get all the density operator describing the field state
  – Present it in terms of the field’s Wigner function:
    • A ‘wavefunction’ in the phase plane (Fresnel plane)
    • A quasiprobability distribution for the complex field amplitude.


Coherent states

• A coherent state with 2.5 photons

Fock states

• $n=0$
Fock states

• \( n=1 \)


F=0.98

Decoherence of Fock states

• Non-classical states are short-lived
  – Rapidly transformed into more classical ones by unavoidable relaxation processes
    • Here: cavity damping \( T_c=0.13 \) s

• Single photon lifetime (at zero temperature)
  • \( \kappa = T_c \) the classical field energy damping time
  • Also applies to coherent states
    – Fock states superpositions produced by classical sources
    – Pointer states of the cavity-environment interaction

• \( |n\rangle \) lifetime: \( T_c/n \)
  – Relaxation time much shorter than the energy lifetime
  – Relaxation time decreases with the size of the state
    • A typical decoherence effect
    • A Fock state is quite similar to the Schrödinger cat!

Lifetime of the \( n \) photon Fock state using past quantum states

• Analyze average time between jumps
  – Fock states lifetime \( T_c/n \)

An impossible feat with forward estimation only due to spurious noise-induced jumps (Brune et al. PRL 101 240402)

T. Rybarczyk et al., PRA 91 062118

T. Rybarczyk et al., PRA 91 062118

F=0.51
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Bohr’s thought experiment on complementarity

• Complementarity (From Einstein-Bohr at the 1927 Solvay congress)
  – Moving slit records the trajectory of the particle in the interferometer
  - Which path information but no fringes
  - Or no which path but fringes
  - Wave and particle are complementary aspects of the quantum object.

Cavity field as a which path detector

• Insert non-resonant cavity inside the interferometer
  – Cavity contains initially a mesoscopic coherent field
  – The two atomic levels produce opposite phase shifts of the cavity field
    \[
    |e\rangle \rightarrow |e\rangle, \quad |g\rangle \rightarrow |g\rangle \tag{1}
    \]
  – Field amplitude is the ‘needle’ of a ‘meter’ pointing towards atomic state
    – Prototype of a quantum measurement
    – Provides a which-path information and should erase the fringes

Action of an atom on a coherent field in the dispersive regime

• Effective Hamiltonian
  \[
  H_{eff} = \hbar \omega \left(\sigma_x (a^\dagger a^2 ) + \frac{1}{2} \right)
  \]
  \[
  U_{eff}(t) = e^{-i\frac{\Phi}{2}} e^{-i\sigma_x a^\dagger a} e^{-i\frac{\Phi}{2}} \quad \Phi = \hbar \omega t.
  \]
• Apply to \(|e, \alpha\rangle\)
  \[
  e^{-i\Phi |e, \alpha\rangle} = e^{-i\frac{\Phi}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-i\sigma_x a^\dagger a} |e, \alpha\rangle
  \]
  \[
  - e^{-i\frac{\Phi}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-i\sigma_x a^\dagger a} |e, \alpha\rangle = |e, e^{-i\Phi}\rangle
  \]
  \[
  |g, \alpha\rangle \rightarrow e^{-i\Phi} |g, \alpha e^{-i\Phi}\rangle
  \]
  \[
  |g, \alpha\rangle \rightarrow |g, \alpha e^{-i\Phi}\rangle
  \]
  – The atom (quantum system) controls the classical phase of the field
  – At the heart of Schrödinger cat states generation
Two limiting cases

• Small phase shift (large $D$) (smaller than quantum phase noise)
  - Field phase almost unchanged
  - No which path information
  - Standard Ramsey fringes

• Large phase shift (small $D$) (larger than quantum phase noise)
  - Cavity fields associated to the two paths distinguishable
  - Unambiguous which path information
  - No Ramsey fringes

Fringes and field state

• An illustration of complementarity

Signal analysis

Fringe signal multiplied by $\langle \alpha e^{i\phi} | \alpha e^{-i\phi} \rangle$

• Modulus $e^{-2\text{Im} \sin \Phi} = e^{-D^2/2}$
  - Contrast reduction

• Phase $2\pi \sin \Phi$
  - Phase shift corresponding to cavity light shifts

Phase leads to a precise (and QND) measurement of the average photon number

Calibration of the cavity field 9.5 (0.1) photons

A laboratory version of the Schrödinger cat

Field state after atomic detection

A coherent superposition of two "classical" states.
Very similar to the Schrödinger cat

Decoherence will transform this superposition into a statistical mixture
Slow relaxation: possible to study the decoherence dynamics
Decoherence caught in the act

An atom to probe field coherence
Quantum interferences involving the cavity state
First atom
Second atom
Two indistinguishable quantum paths to the same final state:
Quantum interference in a two-atom correlation signal
A decoherence study

**Atomic correlation signal**

Decoherence versus size of the cat

---

A simple calculation of a cat’s decoherence

- **A cat in a cavity coupled to a bath of linear oscillators**
  - Linear cavity-bath coupling: a coherent state in the cavity couples to time-dependent coherent fields in the environment modes (no cavity-environment entanglement)
  - A cat disseminates small kittens in the environment

---

A simple calculation of a cat’s decoherence

- **Complete wavefunction at time \( t \):**
  \[
  \exp \left[ -i \sum_{\Phi} \Phi \left( 1 - e^{i \Phi} \right) \right]
  \]
  - Cavity state entangled with environment

- **Remaining cat’s coherence when tracing over the environment**
  \[
  \exp \left[ -2\sum_{\Phi} D \Phi \right] \]
  - Experimental signal: 0.5x real part of this quantity

- **Energy conservation**
  \[
  \sum |\beta(t)|^2 = \pi \left( 1 - e^{-1/T_c} \right)
  \]

---

A simple calculation of a cat’s decoherence

- **Remaining coherence**
  \[
  \exp \left[ -2\sum_{\Phi} D \Phi \right] \]
  - Decoherence time scale
  \[
  T_c / 2n = 2T_c / D
  \]
  \( D \): distance between cat components

- **In terms of Monte Carlo quantum trajectories**
  - Cat switches parity at each photon loss
  - Parity undetermined when one photon lost on the average
Schrödinger cat states

- Even cat
  - $n=3.5$ photons
  - $\zeta=0.37\pi$
  - $D^2=11.8$ photons

- Odd cat

- Statistical mixture of cats (or of coherent states)

A movie of the even cat decoherence

Decoherence time

\[ T_d = 17 \pm 3 \text{ ms} \]


For similar work in circuit QED see Wang et al. PRL 103 200404

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Feedback: a universal technique

- Classical feedback is present in nearly all control systems
  - A SENSOR measures the system's state
  - A CONTROLLER compares the measured quantity with a target value
  - An ACTUTATOR reacts on the system to bring it closer to the target

- Quantum feedback has the same aims for a quantum system
  - Stabilizing a quantum state against decoherence
  - Must face a fundamental difficulty:
    - measurement changes the system state

Two quantum feedback experiments

- Prepare and preserve a Fock state in the cavity
  - Target state: the photon number state \( n \)
- Feedback loop
  - Get information on the cavity state
    - QND quantum sensor atoms sent at 82 µs time interval
  - Estimate cavity state and distance to target
    - Fast real-time computer (ADWin Pro II)
      - A complex computation taking into account all known imperfections
    - Decide upon actuator action
  - Actuator action
    - Drives the cavity state as close as possible to the target
Two experiments

- **Classical actuator**
  - Actuator is a coherent source
  - Displacement of the cavity field
  - Technically simple
  - Not optimal: complex procedure to correct for single photon loss
  - Preparation and protection of Fock states up to $n=4$

- **Quantum actuator**
  - Resonant atoms used to inject/subtract photons
  - More demanding experimentally
  - Faster quantum jumps correction
  - Stabilization of Fock states up to $n=7$


A single trajectory: closed loop

- **Target photon number $n_t=4$**

Feedback for high photon numbers

- Reference coherent state with $n_t$ photons on the average
  - Steady state
  - Stops loop at 140 ms
  - Independent QND estimation of average photon number distribution $P(n)$

- Optimal stop
  - Stops loop when $p(n_t)=0.8$
  - Independent QMD estimation of $P(n)$

- Stabilization of photon numbers up to 7
- Convergence twice as fast as that of the feedback with coherent source
Using feedback to optimize QND measurement

• Send atoms one by one and use previous information to optimize information brought by next atom

• A simple scheme in an ideal setting
  – Assume n<8 (0 through 7 photons)
    – First atom sent in $g$ with $\phi_0=\pi, \phi_r=0$
      • Detected state tells the field parity
        – Detected in $e$ when empty or even photon number
        – Detected in $g$ when odd photon number
      • Atom gives the Least significant bit of photon number
      • Projects the field on a parity eigenstate (cat if initial state coherent)
    – Second atom sent with $\phi_0=\pi/2$
      • Phase $\phi_f$ adjusted to distinguish
        – 0,4 from 2,6 if parity even
        – 1,5 from 3,7 if parity od
      • Atom gives the second bit of the photon number

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  – Towards a circular state quantum simulator

Using feedback to optimize QND measurement

• A simple scheme in an ideal setting
  – Third atom sent with $\phi_0=\pi/4$
    • Ramsey phase set to remove the last ambiguity
    • Atom gives the third bit of the photon number
  – Measurement of photon number from 0 to 7 with 3 atoms
    • Instead of 110

• Straightforward generalization
  – Measurement of photon number from 0 to N-1 with $\log_2(N)$ atoms
  – Optimum set by information theory
  – An optimal quantum digital/analog converter

• Realistic setting
  – Measure photon number from 0 to 7 with ~13 atoms (instead of 110)

Quantum simulation

  – “Nature isn’t classical, dammit, and if you want to make a simulation of nature, you’d better make it quantum mechanical, and by golly it’s a wonderful problem, because it doesn’t look so easy”.

• The Hilbert space dimension problem
  – $N$ spins $1/2$: dimension $2^N$. A quite rapidly growing function!
  – Explicit numerical calculations out of reach of the largest supercomputers as soon as $N>42$ (roughly)

• Approximate numerical methods
  – DMRG, t-DMRG, MPS,….
    • Very efficient for some questions (ground state in 1D up to $N=$few 100)
    • Limited for others (long-term dynamics with large entanglement, many-body localization, quenches….)
Quantum simulation

• Follow Feynman’s precepts
  – Build a quantum machine to simulate the quantum
  • Realize a fully controllable/measurable system with the same
dynamics as the system of interest
  – More efficient than exact classical computations for large
  Hilbert spaces (N>42)

• Digital quantum simulation
  – Build a full-fledged quantum computer
  • Run it to simulate the Hamiltonian of interest
  – Already implemented with ion traps
  – As difficult as a full-fledged quantum computer
  » Many qubits and many gates : “it doesn’t look so easy”

• Analog quantum simulation

Analog quantum simulation

• Many realizations already

Dipole-Dipole interaction between Rydberg atoms

• A long range, strong interaction
  – Early evidence
  – Direct measurement
    Béguin et al PRL 110, 263201

  – Two 60S Rydberg levels
  • Isotropic, repulsive interaction
    \[ V_{dd} = \frac{C_6}{r^6} \]
  • Order of magnitude
    8.8 MHz at 5 μm
    – To be compared with a typical 20 kHz kinetic energy in cold
cloud at 1 μK
Dipole blockade and facilitation

- Laser excitation of a dense cloud of ground state atoms
  - At resonance:
    - Blockade radius determined by the excitation linewidth
  - Above resonance (positive laser detuning)
    - Faciliation radius
    - Excitation of atomic clusters


Quantum simulation with Rydberg atoms

- A more controlled situation
  - Rydberg excitation of atoms in an optical lattice
    - Or dressing of ground states with a Rydberg level
  - A few exciting results
    - Crystallization of Rydberg excitations (Schauss et al. Science 347 1455)
    - Many-body dynamics (Bernien et al., Nature 551, 579)

Quantum simulation with Rydberg atoms

- Two-dimensional quantum Ising models (Labuhn et al., Nature, 534, 667)
  - Limitations
    - Finite lifetime (100 μs for laser accessible states)
    - And blackbody-induced transfers
    - Atomic motion
      - An even more severe limitation to the useful time
      - Reduced but not cancelled by Rydberg dressing of ground states
  - Is it possible to operate with long-lived Rydberg atoms trapped in an optical lattice?
    - Towards a trapped circular Rydberg atom quantum simulator
      - T.L. Nguyen et al. PRX 8 011032

Circular Rydberg atoms

- Non-degenerate with manifold in F and B fields
- Long lifetime
  - 25 ms for 48C. Main decay channel: microwave spontaneous emission on a σ+ transition
- Spontaneous emission inhibition
  - Emission inhibited in a capacitor below cut-off.
    - 2500 s life in a 13 x 2 mm capacitor!
  - Remaining decay channels
    - vdW interaction state mixing
    - Blackbody absorption (0.5 K)
    - Lifetime 60 s
  - Very long lifetime for a pair of interacting 48C atoms at a 5 μm distance
    - Trapping mandatory
Circular states laser trapping

- Circular states can be laser-trapped!  
  - Ponderomotive electron energy:
    - atoms are low-field seekers
    - a large trap
      - ~10 times greater polarizability that of ground state Rubidium at 1 µm wavelength
    - Trapping almost independent of principal quantum number
      - Low trap-induced decoherence
      - Impervious to photoionization
    - severe limitation for low l states
  - Long term trapping
    - 50 s lifetime taking into account Compton scattering and realistic vacuum conditions in a cryogenic environment
    - >1 s lifetime for a 40 atoms chain

A simple trap geometry for a 1-D lattice

- Trapping lasers at 1 µm
- LG mode along Ox (transverse trap)
- Two Gaussian beams at a small angle
  - Longitudinal lattice with an adjustable spacing
    - d = 5 to 7 µm
    - 24 kHz longitudinal oscillation frequency

Circular Rydberg interaction

- Choice of levels
  - Encode spin states on 48C and 50C
    - A repulsive van der Waal interaction ($\propto 1/d^6$) between atoms in the same levels at a distance d
    - A second order spin exchange interaction (48C,50C to 50C, 48C) ($\propto 1/d^6$)
  - Dress the atomic transition with a near-resonant microwave
    - Rabi pulsation $\Omega$, detuning $\Delta$
      - Makes the ground state nontrivial
      - Can be fed in the capacitor in an evanescent mode
  - Realization of the XXZ spin-1/2 chain Hamiltonian

$$\frac{H}{\hbar} = \frac{\Delta}{2} \sum_{j=1}^{N} \sigma^z_j + \frac{\Omega}{2} \sum_{i=1}^{N} \sigma^x_i + J_2 \sum_{j=1}^{N-1} \sigma^x_j \sigma^x_{j+1} + J \sum_{j=1}^{N-1} (\sigma^y_j \sigma^y_{j+1} + \sigma^z_j \sigma^z_{j+1})$$
**A rich phase diagram**

- Within reach of parameters tuning range
  - $\Delta=0$
    - Dimitriev et al., JETP 95, 538
- For realistic atom numbers (MPS for 40 atoms)

**Deterministic chain preparation**
- Van der Waals evaporation
  - LG and `plug' beams trap
  - One weak, one strong
  - Load ~ 100 circular atoms
  - Compress the trap. Atom evaporate above weak plug
  - Classical modelization
    - Final atom determined by trap length
    - Deterministic chain preparation up to ~40
- Effective cooling
  - Final motion amplitude close to ground state one
- Chain state detection
  - Interrupt exchange (48C to 46C, exchange stops)
  - Resume evaporation, routing atoms one by one to a field-ionization detector: measure all $s_z$ observables
  - Additional hard microwave pulse: measure all spin observables.

**Exploring the phase diagram**
- From F to $P_x$ phase and back
- Exact diagonalization calculations
  - 14 atoms
  - $J_z/\Omega=1.6$, $J_z=2.3$ kHz (7 $\mu$m)
  - Including atomic motion
  - Optimized $\Omega$ ramp
  - 77 $\mu$s total duration
- A good observation of the QPT
  - Main observables precisely follow MPS ground state
  - Negligible influence of residual atomic motion
  - High initial state return fidelity (99%)
A team work

- S. Haroche, M. Brune, J.M. Raimond, S. Gleyzes, I. Dotsenko, C. Sayrin

- Cavity QED
  - V. Métillon, F. Assémat

- Rydberg metrology
  - A. Larrouy, R. Richaud, A. Muni, L. Lachaud

- Circular state simulator
  - R. Cortinas, B. Navon, P. Méhaignerie, H. Wu

A team work (1973-2019)

- Serge Haroche
- Michel Gross
- Michel Fabre
- Philippe Goy
- Pierre Pillet
- Jean-Michel Raimond
- Frédéric Bréger
- Francoise Calle
- Jean-Michel Raimond
- Golgi Viard
- Yves Kaluszyk
- Jean Léon
- Michel Steel
- Valérie Lefèvre-Seguin
- Jean Hare
- Jacques Lapujade
- Asaphent Sankar
- André Ruprecht
- Frédéric Bernardin
- Paul Nussenzweig
- Laurent Collot
- Matthias Wiederhöfer
- François Teussier
- Abdessamed Mait
- David Varea
- Yael Tzvi
- Jonathan Knibb
- Nikoia Dubreuil
- Peter Damore
- Ferdinand Schmidt-Kaler
- Jochen Dreyer
- Elia Hagley
- Xavier Mahn
- Christoph Wunderlich

- Gilles Nogues
- Vladimir Ivanov
- Robert Schindler
- Stefano Openghi
- Ann-Rauscherblau
- Wolf van Kämping
- Zefri Ulis
- Patrice Berlet
- Anne Fufuelle
- Rémi-Long
- Sébastien Simon
- Pascale Mazê
- Angel Díaz
- Philippe Houry
- Thibaut Auffèves
- Perival Mitter
- Jack Morley
- Stefan Kühn
- Carla Hermann
- Christine Guehrin
- Thomas Németh
- Cedric Ross
- Julian Bervi
- Ulrich Bula-Hoff
- Andreas Embleton
- Adrian Lupascu
- Jonas Myers
- Igor Czajkowski
- Samuel Delahaye
- Johann Schacht

- Xinping Zou
- Bruno Peaudecerf
- Raul Teixeira
- Shi Liu
- Tade Rymanicki
- Carla Hermann
- Adrian Sargent
- Ashid Farzin
- Eva Deuchene
- Sefal Gerich
- Shui Liang Ngyen
- Mariana Penasco
- Dina Schmico
- Tigrane Cattai
- Frédéric Assémat
- Valérie Melton
- Rodrigo Cortina
- Attila Lemrny
- Paul Meneghini
- Remy Richard
- Hanqin Wu
- King Ravan
- Andrea Muni
- Léa Lachaud
- Yohann Machu