Laboratoire Kastler Brossel
Physique quantique et applications

Field quantization and cavity quantum electrodynamics

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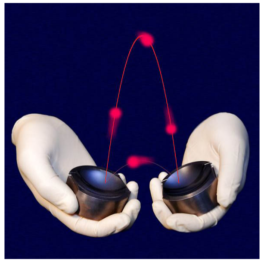
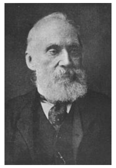


RYSQ ANR QuSCo erc TRENSCRYBE

Atoms and photons have triggered the quantum revolution

- The triumphant classical physics at the end of the 19th century
  - Three pillars
    - Mechanics (Newton): celestial bodies motion, machines…
    - Thermodynamics (Boltzmann). The science of heat and engines
    - Electromagnetism (Maxwell). Electricity, magnetism and light
  - Countless success
- Only two ‘little clouds’ (lord Kelvin) at the interfaces…
    - Ether problem (relativity of the velocity of light)
    - Blackbody radiation (light emission by heated bodies)
    - And also
      - Atomic spectra



- …Which turned into a great storm
  - No understanding of microscopic physics in the ‘classical’ frame

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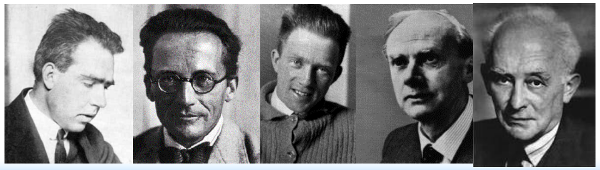
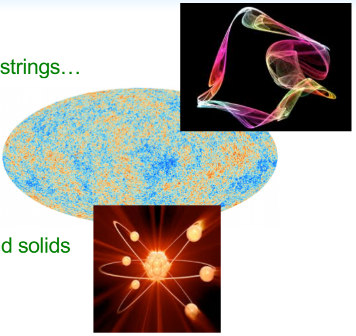
The quantum revolution

- 1926-1930
  - An impressive success
    - Atomic structure finally understood
  - Many counterintuitive features
    - Raises fundamental questions on physics
  - A mandatory interpretation
    - Link between the mathematical framework and the experimental reality.
  - A few individuals have changed our understanding of the world



An unprecedented series of sucess…

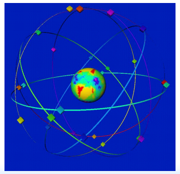


- Huge range of applications
  - From elementary particles and strings…
    - 10^{-35} m to 10^{-15} m
  - … to cosmological structures…
    - 10^{26} m
  - …through atoms, molecules and solids
    - 10^{-10} m
- Extremely precise predictions
  - Agreement between theory and experiment over 12 digits !
- A universal theoretical frame
  - All interactions (but gravity) in a single formalism

…countless applications…

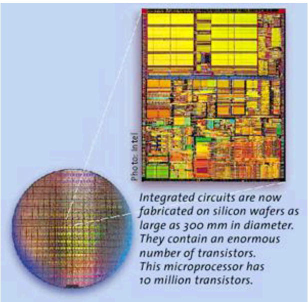
- Atoms and photons
  - Lasers essential for long-haul information transmission in optical fibres i.e. for internet!

  - Ultra-precise atomic clocks
(1s over the age of the Universe)
are at the heart of the GPS system



…countless applications…

- Solid state physics
  - Integrated circuits rely on our quantum understanding of electrical currents in semiconductors (silicium)



Integrated circuits are now fabricated on silicon wafers as large as 300 mm in diameter. They contain an enormous number of transistors. This microprocessor has 10 million transistors.

…countless applications…

- Nuclear magnetic resonance imaging (MRI) is a combination of quantum technologies….

Superconducting magnets

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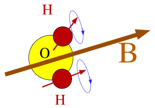
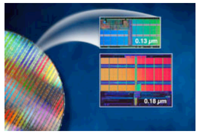


Quantum dance of the Hydrogen nuclei in magnetic fields

+



Data processors for image reconstruction

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…countless applications

- A considerable societal and economic impact
  - Large part of our GDP (40%) results from quantum technologies
    - Also large part of our lifetime expectancy!
  - No information society without the quantum
- An astounding example of the impact of curiosity-driven blue-sky research on the long term
  - Science needs time
    - One century of quantum physics to shape the society
      - Far beyond the 5 years horizon of science granting systems
  - Science needs freedom
    - Investigate exotic problems even though they do not look like a scientific priority at the time

Atoms and light

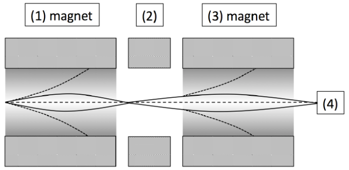
- Have initiated the quantum revolution
- Continued to play a central role in the development of quantum physics
  - On a conceptual level
    - Problems of diverging energy shits in vacuum (Lamb shift) and electron magnetic moment (g-2) led to renormalization techniques, modern quantum electrodynamics and, later, to the standard model
  - On an experimental level
    - Development of increasingly sophisticated methods to manipulate atoms with light (or light with atoms)
    - Periodic technological breakthroughs lead to periodic revivals of AMO physics

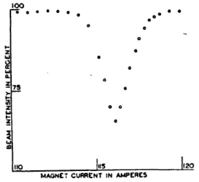
- Let us discuss a short history of these developments

  - (S. Haroche's lectures in Collège de France, 2015 https://www.college-de-france.fr/site/serge-haroche/course-2014-2015.htm)

Rabi 1935

- Inducing rf transitions in an atom

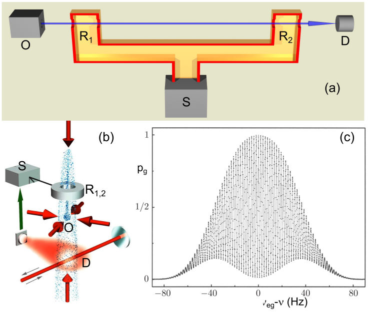


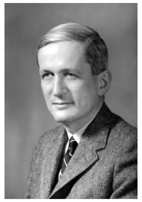


  - First manipulation of an atomic spin by radiofrequencies
  - Still a key ingredient in all AMO experiments

Ramsey 1949

- Separated oscillatory field method

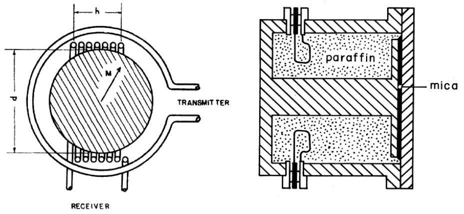




  - A first example of quantum interferometry
  - A powerful spectroscopic tool
  - Key ingredient in atomic clocks
    - Much more on Ramsey fringes in these lecture

Bloch and Purcell 1946

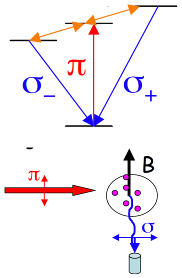
- Nuclear magnetic resonance

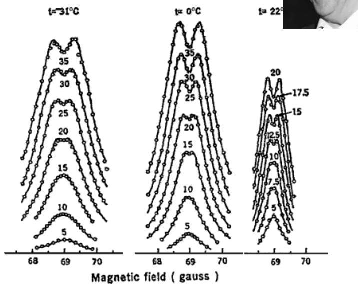


  - RF Manipulation of nuclear spins in magnetic fields
  - A key technology for chemical analysis
    - and for medical imaging

Brossel and Bitter 1950

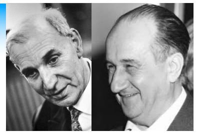
- Double resonance



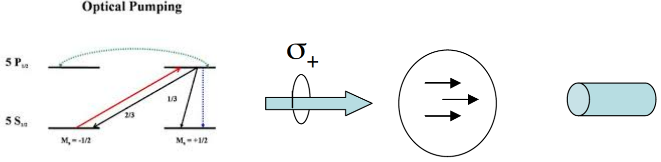


  - Optical detection of radiofrequency transitions
    - Observation of multiple quanta transitions
    - A harbinger of optical pumping

Kastler and Brossel 1950



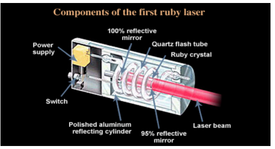
- Optical pumping

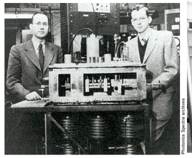


  - Manipulation of the atoms angular momentum by the angular momentum of light
    - Creation of out-of-equilibrium situations
    - A powerful tool for atomic physics
    - Suggestion to manipulate also the external degrees of freedom
      - Effet lumino-frigorique

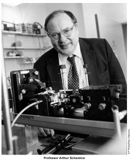
Townes, Gordon, Schawlow, Mainman, Javan…1960

- Masers and lasers

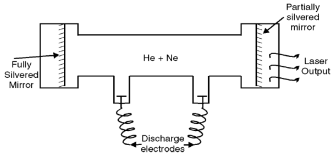




Charles Townes (left) and James P. Gordon

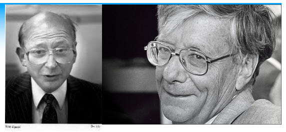


Professor Arthur Schawlow



  - A technological revolution in AMO physics
    - Infinitely powerful, infinitely sharp…
    - Allowed us also to enter digital communication age

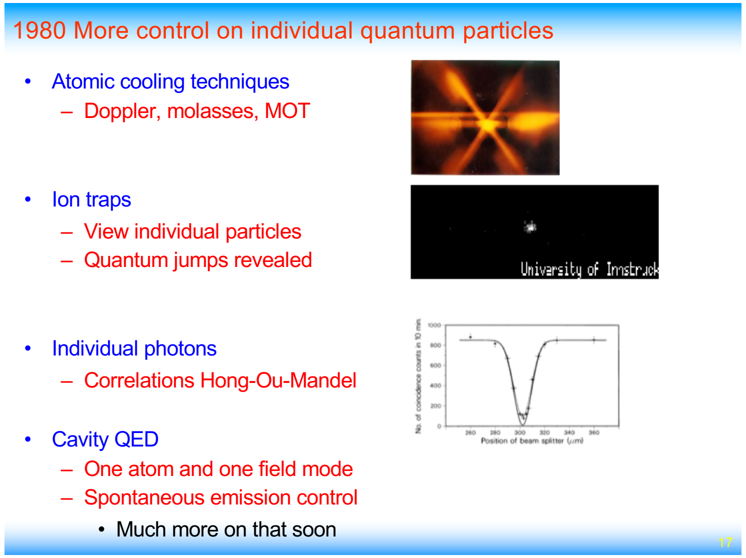
Non linear phenomena… 1960-1970



- Franken, Bloembergen
  - Non linear optics
    - Harmonic generation, frequency mixing…

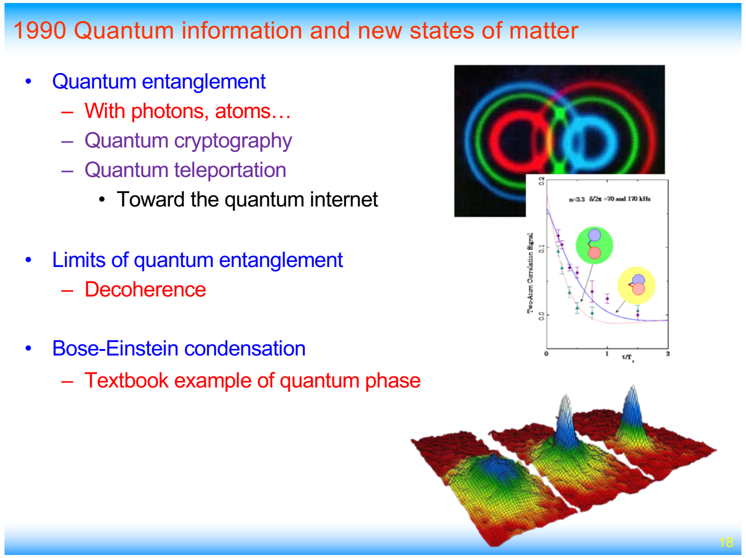
- Bordé, Hall
  - Saturated absorption
    - Doppler-free spectroscopy

- Grynberg, Cagnac, Chebotaiev
  - Two-photon doppler-free spectroscopy
    - High resolution spectroscopy of narrow atomic lines
    - Toward Hydrogen spectroscopy



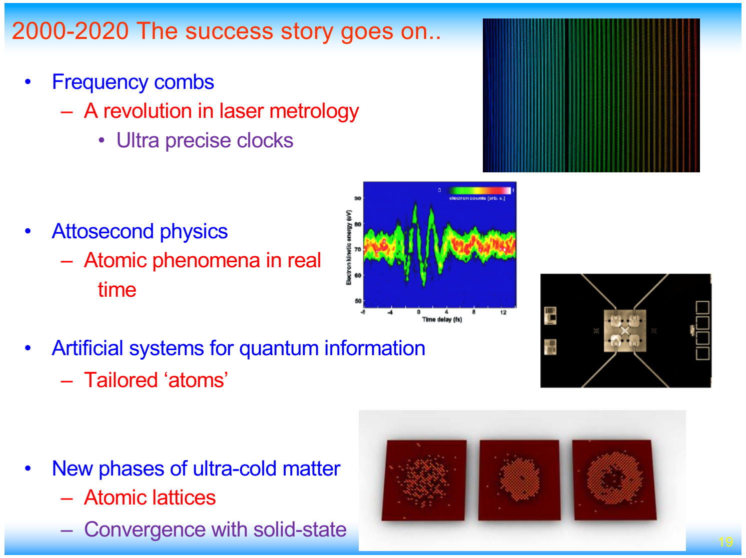
1980 More control on individual quantum particles

- Atomic cooling techniques
  - Doppler, molasses, MOT
- Ion traps
  - View individual particles
  - Quantum jumps revealed
- Individual photons
  - Correlations Hong-Ou-Mandel
- Cavity QED
  - One atom and one field mode
  - Spontaneous emission control
    - Much more on that soon



1990 Quantum information and new states of matter

- Quantum entanglement
  - With photons, atoms…
  - Quantum cryptography
  - Quantum teleportation
    - Toward the quantum internet
- Limits of quantum entanglement
  - Decoherence
- Bose-Einstein condensation
  - Textbook example of quantum phase



2000-2020 The success story goes on..

- Frequency combs
  - A revolution in laser metrology
    - Ultra precise clocks
- Attosecond physics
  - Atomic phenomena in real time
- Artificial systems for quantum information
  - Tailored 'atoms'
- New phases of ultra-cold matter
  - Atomic lattices
  - Convergence with solid-state

9 or 10 orders of magnitude gain in 50 years on:

- Clock precision
  - From 10^{-8} for quartz clocks to 10^{-18} for lattice clocks
- Spectroscopic resolution
  - 10^{-5} to 10^{-15} on Hydrogen spectroscopy
- Atomic detection efficiency
  - From 10^{10} atoms to 1 atom
- Atomic temperatures
  - From 1 K to sub-nK
- Laser pulse duration
  - From 10 ps (modelock lasers) to 100 as
- Laser peak intensity

This school

- Interaction of light with cold atoms
  - A key ingredient in modern AMO and quantum optics
    - A prerequisite for any research in this domain
  - A paradigmatic example of fundamental quantum process
    - An opportunity to wonder on the quantum
  - Opens fascinating routes for fundamental physics and applications
    - New states of matter
    - Quantum communication
    - Quantum computing
    - Quantum simulation
    - Quantum metrology

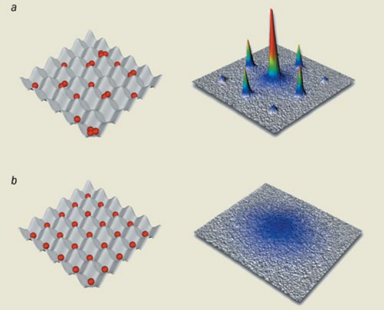
The program

- **Jean-Michel Raimond**, LKB, SU: *Introductory lecture, Field quantization, Dressed states and Cavity QED*
- **Mikhail Baranov**, University of Innsbruck: *Quantum gases and superfluidity*
- **Hélène Perrin**, LPL, CNRS, University Paris 13: *Optical lattices*
- **Philippe Verkerk**, PhLAM, CNRS, University of Lille: *Laser cooling*
- **Ana Asenjo Garcia,** Columbia University: *Collective phenomena in light-matter interfaces*
- **Antoine Browaeys,** Institut d'Optique Graduate School, CNRS, Université Paris Saclay: *Dipole - dipole interaction*
- **Robin Kaiser,** INPHYNI, CNRS, UCA: *Interaction and disorder*
- **Ekkehard Peik,** PTB: *Applications: from high precision measurements to metrology*
- **Jook Walraven,** University of Amsterdam: *Ultracold collisions*

Mikhail Baranov: *Quantum gases and superfluidity*

Hélène Perrin: **Optical lattices** (3 lectures)

- **Lecture 1**: Band structure in a periodic potential: Bloch functions, energy bands
- **Lecture 2**: Dynamics in the lattice: time-of-flight, adiabatic switching and band mapping, Bloch oscillations
- **Lecture 3**: Tight-binding limit: from Wannier functions to the Bose-Hubbard Hamiltonian. Mott transition



Philippe Verkerk: Laser cooling (4 lectures)

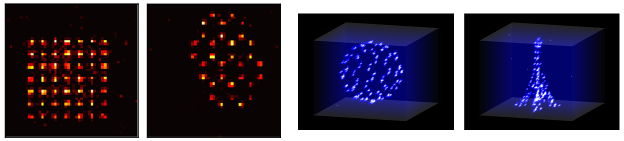
- **Lecture 1**: Two-level atoms; light forces
    - Two-level atom
    - Semi-classical description
    - Broad-band limit
    - Force operator
- **Lecture 2**: Doppler cooling and the magneto-optical trap
- **Lecture 3**: Sub-Doppler cooling
- **Lecture 4**: Beyond the simple MOT; non-linear objects

Ana Asenjo-Garcia:
Collective phenomena in light-matter interfaces

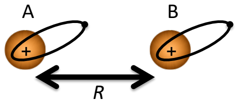
- 1- Atom-light interaction as a spin model
    - quantization of the electromagnetic field using Green's functions
    - meaning of Born, Markov, and rotating wave approximations
    - effective spin model for atom-atom interactions
- 2- Atom arrays as light-matter interfaces
    - collective phenomena: super- and sub-radiance
    - physics of subradiant states in 1D and 2D ordered arrays
    - classical vs quantum interference
- 3- Atom-atom interactions in non-conventional baths
    - modifying interactions through nanophotonics
    - waveguide QED: atoms and other quantum emitters close to fibers and photonic
    - crystals
    - applications for quantum information science

Dipole interactions between atoms: QIP and many-body physics

Antoine Browaeys



Rydberg interactions



Van der Waals $\frac{C_6}{R^6}$

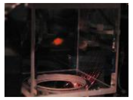
resonant $\frac{C_3}{R^3}$

1. Dipole-dipole interactions between atoms
2. Many-body physics with synthetic matter
3. Quantum information processing

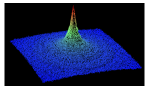
Robin Kaiser : Interaction and disorder :

Light scattering by cold atoms / Localization and Cooperative Scattering

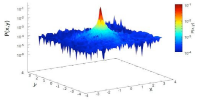
Lecture 1 : Multiple Scattering of Light in Cold Atoms
Steady state results : Ohm's law for photons
Time dependent scattering : radiation trapping
+ Numerical random walk simulations



Lecture 2 : Interference Effects in Light Scattering by Cold Atoms
Coherent backscattering of light by cold atoms
+Numerical simulations with weak localization corrections
'Single Photon' Dicke super- and subradiance
+Numerical simulations with coupled dipoles



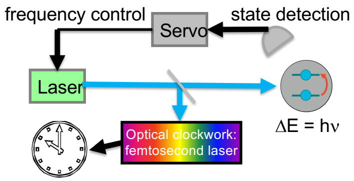
Lecture 3 : Anderson Localisation of Light
Anderson lattice model
Effective Hamiltonian approach
Scalar vs vectorial light : red light for Anderson localization
Outlook : towards localization of light in cold atoms



Link to Mathlab codes :
http://www.kaiserlux.de/coldatoms/LesHouches2019Kaiser.html

Applications: from high precision measurements to metrology

Scheme of an
optical atomic clock:



- High resolution spectroscopy of cold trapped atoms and ions
- Atomic clocks in the microwave and optical domain
- How to measure and quantify uncertainty and instability
- Methods and techniques: Ramsey spectroscopy schemes; laser frequency stabilisation, femtosecond frequency combs
- Frequency comparisons as tests of fundamental principles of physics

Ekkehard Peik, PTB, Braunschweig, Germany

Jook Walraven, University of Amsterdam: *Ultracold collisions*

1. Relative motion of interacting particles I
   - model potentials: range, phase shift, scattering length
2. Relative motion of interacting particles II
   - model potentials: effective range and s-wave resonance
   - generalization to arbitrary short-range potentials
3. Scattering of interacting particles
   - scattering amplitude and cross section
   - distinguishable versus identical particles
4. Scattering of particles with internal structure (atoms)
5. Interaction tuning with magnetic Feshbach resonances

This lecture: Field quantization and CQED

- A basic introduction to quantum atom-field interaction
  - Field quantization
  - Atom-field interaction
  - Cavity quantum electrodynamics

Bibliography

- All quantum optics textbooks
  - In particular:
    - *Cohen-Tannoudji, Dupont-Roc and Grynberg, An introduction to quantum electrodynamics and Photons and atoms, Wiley, 1992;*
    - *Cohen-Tannoudji and Guery Odelin Advances in atomic physics: an overview, World Scientic 2012*
    - *Schleich, Quantum optics in phase space, Wiley 2000;*
    - *Vogel, Welsch and Wallentowitz, Quantum optics an introduction, Wiley 2001*
    - *Meystre and Sargent Elements of quantum optics, Springer 1999*
    - *Barnett and Radmore Methods in theoretical quantum optics, OUP, 1997*
    - *Scully and Zubairy Quantum optics, 1997*
    - *Loudon Quantum theory of light, OUP 1983.*

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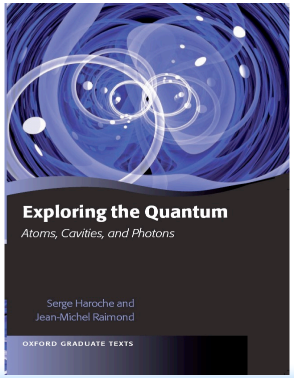
  - http://www.lkb.upmc.fr/cqed/teaching/teachingjmr/



- S. Haroche and J.M. Raimond
  - Exploring the quantum:
    - atoms, cavities and photons
  - Oxford Univ. Press 2006
  - And many references therein

Let us start and quantize the field

Field quantization and cavity QED

Field quantization

J.M. Raimond

September 23, 2019

Outline

1. Field eigenmodes

Outline

1. Field eigenmodes
2. Field quantization, Fock states

Outline

1. Field eigenmodes
2. Field quantization, Fock states
3. Other field quantum states

Outline

1. Field eigenmodes
2. Field quantization, Fock states
3. Other field quantum states
4. Field relaxation

Objective

To quantify the field, we must identify a set of orthogonal modes, the relevant dynamical variables and quantify them according to the ‘canonical’ quantization procedure. The main technical difficulty in field quantization is thus a classical electromagnetism calculation.

Eigenmodes

Positive frequency fields

Temporal Fourier transform of electric field

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\,d\omega \qquad (1)$$

Since **E** is a real field,

$$\widetilde{\mathbf{E}}^*(\mathbf{r},\omega) = \widetilde{\mathbf{E}}(\mathbf{r},-\omega) \qquad (2)$$

Positive frequency field

$$\mathbf{E}^+(\mathbf{r},t) = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} \widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\,d\omega \qquad (3)$$

Negative frequency field

$$\mathbf{E}^-(\mathbf{r},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} \widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\,d\omega = \left(\mathbf{E}^+(\mathbf{r},t)\right)^* \qquad (4)$$

Hence

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}^+(\mathbf{r},t) + \mathbf{E}^-(\mathbf{r},t) \qquad (5)$$

Eigenmodes

Eigenmodes basis

Virtual quantization ‘Box’ of limiting conditions with a total volume $\mathcal{V}$.
Orthogonal basis for the solutions of Maxwell equations for the electric field (a Hilbert space)

$$\mathbf{f}_\ell(\mathbf{r})e^{-i\omega_\ell t} \qquad (6)$$

where the dimensionless amplitude $\mathbf{f}_\ell$ is divergence-free and obeys the Helmholtz equation:

$$\Delta\mathbf{f}_\ell + \frac{\omega_\ell^2}{c^2}\mathbf{f}_\ell = 0 \qquad (7)$$

Orthogonality:

$$\int_{\mathcal{V}} d^3\mathbf{r}\,\mathbf{f}_\ell^*(\mathbf{r})\cdot\mathbf{f}_{\ell'}(\mathbf{r}) = \delta_{\ell,\ell'}\mathcal{V} \qquad (8)$$

Normalization:

$$\int_{\mathcal{V}} d^3\mathbf{r}\,|\mathbf{f}_\ell(\mathbf{r})|^2 = \mathcal{V} \qquad (9)$$

Eigenmodes

Eigenmodes basis

Note: we will use a slightly different normalization condition for a physical quantization box (cavity QED), based on the geometry of the single mode of interest.

Eigenmodes

Eigenmodes basis

Expand the positive frequency field on this basis

$$\mathbf{E}^+(\mathbf{r},t) = \sum_\ell \mathcal{E}_\ell(t)\mathbf{f}_\ell(\mathbf{r}) \tag{10}$$

where

$$\mathcal{E}_\ell(t) = \frac{1}{\mathcal{V}}\int \mathbf{E}^+(\mathbf{r},t)\cdot\mathbf{f}_\ell^*(\mathbf{r})\,d^3\mathbf{r} \tag{11}$$

The amplitude is obviously a harmonic function of time

$$\mathcal{E}_\ell(t) = \mathcal{E}_\ell(0)e^{-i\omega_\ell t} \tag{12}$$

Finally

$$\mathbf{E}^+(\mathbf{r},t) = \sum_\ell \mathcal{E}_\ell(0)e^{-i\omega_\ell t}\mathbf{f}_\ell(\mathbf{r}) \tag{13}$$

Eigenmodes

Plane-wave basis

- A simple basis for a rectangular box and periodic boundaries.
- Set of plane waves with $\mathbf{k}_\mathbf{n} = (k_x, k_y, k_z) = (n_x 2\pi/L_x, n_y 2\pi/L_y, n_z 2\pi/L_z)$, where the ns are positive or negative (not all equal to zero).
- For each $\mathbf{n} = (n_x, n_y, n_z)$, two orthogonal linear polarizations $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$, perpendicular to $\mathbf{k}$: $\boldsymbol{\epsilon}_1 \times \boldsymbol{\epsilon}_2 = \mathbf{u}_\mathbf{k}$.
- Basis

$$\mathbf{f}_\ell(\mathbf{r}) = \boldsymbol{\epsilon}_\ell e^{i\mathbf{k}_\ell\cdot\mathbf{r}} \tag{14}$$

  with $\ell = (n_x, n_y, n_z, \epsilon)$
- Another choice: circular polarization basis

$$\boldsymbol{\epsilon}_\pm = \frac{\boldsymbol{\epsilon}_1 \pm i\boldsymbol{\epsilon}_2}{\sqrt{2}} \tag{15}$$

$$\boldsymbol{\epsilon}_+ \times \boldsymbol{\epsilon}_- = -i\mathbf{u}_\mathbf{k} \tag{16}$$

Normal variables

Potential vector

Choose a simple set of dynamical variables. The potential vector $\mathbf{A}$ is divergence-free in the Coulomb gauge $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$ and $\mathbf{E} = -\partial\mathbf{A}/\partial t$.
$\mathbf{A}$ can be expanded on the same basis as $\mathbf{E}$ (same limiting conditions)

$$\mathbf{A}^+(\mathbf{r},t) = \sum_\ell \mathcal{A}_\ell(t)\mathbf{f}_\ell(\mathbf{r}) \tag{17}$$

Choose $\mathcal{A}_\ell(t)$ (harmonic functions of time) as the normal variables and separate real and imaginary parts

$$\mathcal{A}_\ell(t) = \mathcal{A}_\ell(0)e^{-i\omega t} = \frac{1}{2\sqrt{\epsilon_0\omega_\ell\mathcal{V}}}\left[x_\ell(t) + ip_\ell(t)\right] \ , \tag{18}$$

where we have introduced a normalization factor simplifying the final form of the field energy/hamiltonian in terms of the normal variables.
Note that x_ℓ and p_ℓ have the dimension of the square root of an action...

Normal variables

All fields

From $\mathbf{E}^+ = -\partial \mathbf{A}^+/\partial t$

$$\mathcal{E}_\ell(t) = -\frac{d\mathcal{A}_\ell}{dt} = i\omega_\ell \mathcal{A}_\ell \tag{19}$$

and hence

$$\mathbf{E}^+(\mathbf{r},t) = \sum_\ell i\omega_\ell \mathcal{A}_\ell(t)\mathbf{f}_\ell(\mathbf{r}) \tag{20}$$

Magnetic field:

$$\mathbf{B}^+(\mathbf{r},t) = \sum_\ell \mathcal{A}_\ell(t)\mathbf{h}_\ell(\mathbf{r}) \tag{21}$$

where

$$\mathbf{h}_\ell(\mathbf{r}) = \boldsymbol{\nabla} \times \mathbf{f}_\ell(\mathbf{r}) \tag{22}$$

Field energy

The total field energy

$$H = \frac{\epsilon_0}{2}\int E^2 + \frac{1}{2\mu_0}\int B^2 \tag{23}$$

must be written in terms of real fields

$$\mathbf{E} = 2\mathrm{Re}\,\mathbf{E}^+ = 2\mathrm{Re}\sum_\ell i\omega_\ell\mathcal{A}_\ell\mathbf{f}_\ell \tag{24}$$

Taking into account the modes orthogonality conditions

$$H = \sum_\ell H_\ell \tag{25}$$

Remains to evaluate energy of one given mode. Drop index ℓ for the time being.

Field energy

Electric energy

Real field

$$\mathbf{E} = i\omega\left[\mathcal{A}\mathbf{f} - \mathcal{A}^*\mathbf{f}^*\right] \tag{26}$$

or

$$\mathbf{E} = -\sqrt{\frac{\omega}{\epsilon_0\mathcal{V}}}\left[x\mathbf{f}'' + p\mathbf{f}'\right] \tag{27}$$

with

$$\mathbf{f} = \mathbf{f}' + i\mathbf{f}'' \tag{28}$$

$$H_e = \frac{\omega}{2\mathcal{V}}\left[x^2\int(\mathbf{f}'')^2 + p^2\int(\mathbf{f}')^2 + 2xp\int\mathbf{f}'\cdot\mathbf{f}''\right] \tag{29}$$

Field energy

Magnetic energy

With

$$\mathbf{B} = \mathcal{A}\mathbf{h} + \mathcal{A}^*\mathbf{h}^* = \frac{1}{\sqrt{\omega\epsilon_0\mathcal{V}}}\left[x\mathbf{h}' - 2p\mathbf{h}''\right] \tag{30}$$

we get

$$H_b = \frac{c^2}{2\omega\mathcal{V}}\left[x^2\int(\mathbf{h}')^2 + p^2\int(\mathbf{h}'')^2 - 2xp\int\mathbf{h}'\cdot\mathbf{h}''\right] \tag{31}$$

Similar, but not obviously equal, to the electric energy.

Field energy

Comparing the energies

Let us start with the integral of $(\mathbf{h}')^2$, with $\mathbf{h} = \boldsymbol{\nabla}\times\mathbf{f}$. Using

$$\boldsymbol{\nabla}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\boldsymbol{\nabla}\times\mathbf{a}) - \mathbf{a}\cdot(\boldsymbol{\nabla}\times\mathbf{b}) \tag{32}$$

we can write

$$\boldsymbol{\nabla}\cdot[\mathbf{f}'\times(\boldsymbol{\nabla}\times\mathbf{f}')] = (\boldsymbol{\nabla}\times\mathbf{f}')^2 - \mathbf{f}'\cdot(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{f}') \tag{33}$$

Using that these fields are divergence-free and with Helmoltz equation:

$$\boldsymbol{\nabla}\cdot[\mathbf{f}'\times(\boldsymbol{\nabla}\times\mathbf{f}')] = (\mathbf{h}')^2 - \frac{\omega^2}{c^2}(\mathbf{f}')^2 \tag{34}$$

Integrating over space:

$$\int(\mathbf{h}')^2 = \frac{\omega^2}{c^2}\int(\mathbf{f}')^2 \tag{35}$$

Similarly

$$\int(\mathbf{h}'')^2 = \frac{\omega^2}{c^2}\int(\mathbf{f}'')^2 \tag{36}$$

Field energy

Comparing the energies

Let us examine $\int \mathbf{h}'\cdot\mathbf{h}''$. With

$$\boldsymbol{\nabla}\cdot[\mathbf{f}'\times(\boldsymbol{\nabla}\times\mathbf{f}'')] = (\boldsymbol{\nabla}\times\mathbf{f})\cdot(\boldsymbol{\nabla}\times\mathbf{f}'') - \mathbf{f}'\cdot(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{f}'') \tag{37}$$

$$\int\mathbf{h}'\cdot\mathbf{h}'' = \frac{\omega^2}{c^2}\int\mathbf{f}'\cdot\mathbf{f}'' \tag{38}$$

Hence

$$H_b = \frac{\omega}{2\mathcal{V}}\left[x^2\int(\mathbf{f}')^2 + p^2\int(\mathbf{f}'')^2 - 2xp\int\mathbf{f}'\cdot\mathbf{f}''\right] \tag{39}$$

Using

$$H_e = \frac{\omega}{2\mathcal{V}}\left[x^2\int(\mathbf{f}'')^2 + p^2\int(\mathbf{f}')^2 + 2xp\int\mathbf{f}'\cdot\mathbf{f}''\right] \tag{40}$$

$$\int(\mathbf{f}')^2 + \int(\mathbf{f}'')^2 = \mathcal{V} \tag{41}$$

Field energy

Final result fo a single mode

We get finally

$$H = \frac{\omega}{2}\left[x^2 + p^2\right]\ , \tag{42}$$

the Hamilton function of a one-dimensional harmonic oscillator.
Note that x and p are canonically conjugate variables since

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = \omega p \tag{43}$$

as expected for an $\exp(-i\omega t)$ dependence of $\mathcal{A}(t)$. With p, we get

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} \qquad \text{and} \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial x} \tag{44}$$

as required for canonically conjugate variables, suitable for the quantization procedure.

Field energy

Total energy

The total energy of the radiation field is thus

$$H = \sum_\ell H_\ell = \sum_\ell \frac{\omega_\ell}{2}\left[x_\ell^2 + p_\ell^2\right]\ . \tag{45}$$

Field momentum

Total momentum

Density of momentum proportional to the Poynting vector

$$\mathbf{g} = \frac{\mathbf{\Pi}}{c^2} \qquad \text{with} \qquad \mathbf{\Pi} = \frac{\mathbf{E}\times\mathbf{B}}{\mu_0} \tag{46}$$

The plane wave mode basis is most convenient to describe the momentum

$$\mathbf{E}^+(\mathbf{r},t) = \sum_\ell \mathbf{E}_\ell^+ = \sum_\ell i\omega_\ell \mathcal{A}_\ell(t)\boldsymbol{\epsilon}_\ell e^{i\mathbf{k}_\ell\cdot\mathbf{r}} \tag{47}$$

and

$$\mathbf{B}^+(\mathbf{r},t) = \sum_\ell \mathbf{B}_\ell^+ = \sum_\ell \mathcal{A}_\ell(t)(i\mathbf{k}_\ell\times\boldsymbol{\epsilon}_\ell)e^{i\mathbf{k}_\ell\cdot\mathbf{r}} \tag{48}$$

Field momentum

Total momentum

Using orthogonalities of modes and polarizations

$$\mathbf{P} = \sum_\ell \mathbf{P}_\ell \tag{49}$$

with

$$\mathbf{P}_\ell = \epsilon_0 \int (\mathbf{E}_\ell^+ + \mathbf{E}_\ell^-)\times(\mathbf{B}_\ell^+ + \mathbf{B}_\ell^-) \tag{50}$$

After a painful calculation

Total momentum

$$\mathbf{P} = \frac{1}{2}\sum_\ell |x_\ell + ip_\ell|^2\mathbf{k}_\ell \tag{51}$$

with a clear interpretation.

Field quantization

The field is a collection of independent harmonic oscillators quantified independently, using the Dirac approach.
The conjugate classical variables x and p (we drop the index ℓ in this section) are replaced by two operators X and P (position and momentum operators, with the dimension of the square root of an action) acting in an infinite dimension Hilbert space, with the commutation rule:

$$[X,P] = i\hbar \tag{52}$$

Field quantization

Annihilation and creation operators

We define the reduced quadratures by:

$$X_0 = \frac{X}{\sqrt{2\hbar}} \qquad \text{and} \qquad P_0 = \frac{P}{\sqrt{2\hbar}} \tag{53}$$

With these definitions

$$[X_0, P_0] = \frac{i}{2} \tag{54}$$

We then define the creation and annihilation operators

$$a = X_0 + iP_0 \ , \qquad a^\dagger = X_0 - iP_0 \ , \qquad X_0 = \frac{a+a^\dagger}{2} \ , \qquad P_0 = i\frac{a^\dagger - a}{2} \tag{55}$$

and we get

$$\left[a, a^\dagger\right] = \mathbb{1} \tag{56}$$

Field quantization

Hamiltonian

From the classical field energy, we get the quantum field Hamiltonian

$$H = \frac{\omega}{2}(X^2 + P^2) = \hbar\omega(X_0^2 + P_0^2) \qquad (57)$$

or

$$H = \frac{\hbar\omega}{4}\left[(a + a^\dagger)^2 - (a^\dagger - a)^2\right] \qquad (58)$$

Normal order Hamiltonian

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \qquad (59)$$

whose diagonaization is described in all textbooks.

Field quantization

Number operator

$$N = a^\dagger a \qquad (60)$$

Commutation relations:

$$[N, a] = -a \quad \text{and} \quad \left[N, a^\dagger\right] = a^\dagger \qquad (61)$$

Eingenvalues: all positive integers, with non-degenerate eigenstates

$$N\left|n\right\rangle = n\left|n\right\rangle \;, \qquad (62)$$

Hence, the eigenergies are

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \qquad (63)$$

Ground state: ‘vacuum’, $\left|0\right\rangle$, energy $\hbar\omega/2$

Field quantization

Fock states

$\left|n\right\rangle$ are the ‘photon number states’ with the orthogonality relation

$$\left\langle n \,|p\right\rangle = \delta_{n,p} \qquad (64)$$

Annihilation of photons:

$$a\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle \qquad (65)$$

with

$$a\left|0\right\rangle = 0 \qquad (66)$$

Similarly

$$a^\dagger\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle \qquad (67)$$

Hence

$$\left|n\right\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\left|0\right\rangle \qquad (68)$$

Fock states

A basis of the Hilbert space:

- Pure state $\left|\Psi\right\rangle = \sum_n c_n\left|n\right\rangle$

Photon number distribution

$$p_n = |c_n|^2 \qquad (69)$$

Mean and variance of photon number

$$\overline{n} = \sum_n np_n \qquad \Delta N^2 = \left\langle N^2\right\rangle - \left\langle N\right\rangle^2 = \sum_n (n - \overline{n})^2 p_n \qquad (70)$$

- Statistical mixtures

$$\rho = \sum_{n,p}\rho_{np}\left|n\right\rangle\left\langle p\right| \qquad (71)$$

Photon number distribution

$$\rho_{nn} = p_n \qquad (72)$$

Fock states

Wavefunctions

Basis of eigenstates of the quadratures:

$$X_0 \left| x \right\rangle = x \left| x \right\rangle \qquad \text{and} \qquad P_0 \left| p \right\rangle = p \left| p \right\rangle \tag{73}$$

Wavefunctions:

$$\Psi(x) = \left\langle x \right| \Psi \rangle \tag{74}$$

For the vacuum:

$$\Psi_0(x) = \left(\frac{2}{\pi}\right)^{1/4} e^{-x^2} \tag{75}$$

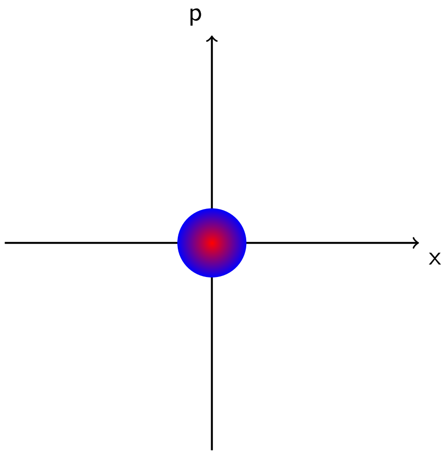
Also in the $\left| p \right\rangle$ representation:

$$\widetilde{\Psi}_0(p) = \left(\frac{2}{\pi}\right)^{1/4} e^{-p^2} \tag{76}$$

Suggests a pictorial representation of the vacuum as a small circle in phase plane.

Fock states

Vacuum state pictorial representation



Fock states

Wavefunctions

For the Fock state $\left| n \right\rangle$:

$$\Psi_n(x) = \left(\frac{2}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} e^{-x^2} H_n(x\sqrt{2}) \tag{77}$$

where H_n is the nth Hermite polynomial defined by

$$H_n(u) = (-1)^n e^{u^2} \frac{d^n}{du^n} e^{-u^2} \tag{78}$$

These wavefunctions have n nodes and a a parity $(-1)^n$

Field operators

All modes

$$H \left| n_1, \ldots, n_\ell \ldots \right\rangle = E_n \left| n_1, \ldots, n_\ell \ldots \right\rangle \tag{79}$$

with

$$E_n = \sum_\ell \left(n_\ell \hbar\omega_\ell + \frac{\hbar\omega_\ell}{2} \right) \tag{80}$$

and

$$\left| n_1, \ldots, n_\ell \ldots \right\rangle = \prod_\ell \frac{(a_\ell^\dagger)^{n_\ell}}{\sqrt{n_\ell !}} \left| 0 \right\rangle \tag{81}$$

Note that the vacuum state has an infinite energy!

Field operators

Vector potential operator

Classical normal variables:

$$\mathcal{A}_\ell = \frac{1}{2\sqrt{\epsilon_0\omega\mathcal{V}}}(x_\ell + ip_\ell) \tag{82}$$

Corresponding quantum operators

$$A_\ell = \frac{1}{2\sqrt{\epsilon_0\omega_\ell\mathcal{V}}}(X_\ell + iP_\ell) = \sqrt{\frac{\hbar}{2\epsilon_0\omega_\ell\mathcal{V}}}a_\ell \tag{83}$$

Positive frequency vector potential

$$\mathbf{A}^+(\mathbf{r}) = \sum_\ell \sqrt{\frac{\hbar}{2\epsilon_0\omega_\ell\mathcal{V}}}a_\ell\mathbf{f}_\ell(\mathbf{r}) \tag{84}$$

Hermitian vector potential:

$$\mathbf{A}(\mathbf{r}) = \sum_\ell \sqrt{\frac{\hbar}{2\epsilon_0\omega_\ell\mathcal{V}}}\left(a_\ell\mathbf{f}_\ell(\mathbf{r}) + a_\ell^\dagger\mathbf{f}_\ell^*(\mathbf{r})\right) \tag{85}$$

Field operators

Electric and magnetic field operators

The hermitian electric field is similarly:

$$\mathbf{E}(\mathbf{r}) = i\sum_\ell \mathcal{E}_\ell\left(a_\ell\mathbf{f}_\ell(\mathbf{r}) - a_\ell^\dagger\mathbf{f}_\ell^*(\mathbf{r})\right) \tag{86}$$

where we define the ‘field per photon in mode ℓ’ by

$$\mathcal{E}_\ell = \sqrt{\frac{\hbar\omega_\ell}{2\epsilon_0\mathcal{V}}} \tag{87}$$

Similarly,

$$\mathbf{B}(\mathbf{r}) = \sum_\ell \sqrt{\frac{\hbar}{2\epsilon_0\omega_\ell\mathcal{V}}}\left(a_\ell\mathbf{h}_\ell(\mathbf{r}) + a_\ell^\dagger\mathbf{h}_\ell^*(\mathbf{r})\right) \tag{88}$$

with $\mathbf{h}_\ell = \boldsymbol{\nabla}\times\mathbf{f}_\ell$

Field operators

Plane wave mode basis

$$\mathbf{A}^+(\mathbf{r}) = \sum_\ell \sqrt{\frac{\hbar}{2\epsilon_0\omega_\ell\mathcal{V}}}a_\ell\boldsymbol{\epsilon}_\ell e^{i\mathbf{k}_\ell\cdot\mathbf{r}} \tag{89}$$

$$\mathbf{E}^+(\mathbf{r}) = i\sum_\ell \mathcal{E}_\ell a_\ell\boldsymbol{\epsilon}_\ell e^{i\mathbf{k}_\ell\cdot\mathbf{r}} \tag{90}$$

$$\mathbf{B}^+(\mathbf{r}) = \sum_\ell \sqrt{\frac{\hbar}{2\epsilon_0\omega_\ell\mathcal{V}}}a_\ell(i\mathbf{k}_\ell\times\boldsymbol{\epsilon}_\ell)e^{i\mathbf{k}_\ell\cdot\mathbf{r}} \tag{91}$$

Field operators

Heisenberg picture

Evolution of annihilation operator

$$i\hbar\frac{da_H}{dt} = [a_H, H] \qquad \text{i.e.} \qquad \frac{da_H}{dt} = -i\omega a_H \tag{92}$$

whose immediate solution is

$$a_H(t) = a_H(0)e^{-i\omega t} = ae^{-i\omega t} \tag{93}$$

Same harmonic evolution as the classical normal variables.

Field operators

Momentum

Total momentum by replacing $|x_\ell + ip_\ell|^2$ in the classical expression by $(x_\ell - ip_\ell)(x_\ell + ip_\ell)$ and $x_\ell + ip_\ell$ by $a_\ell\sqrt{2\hbar}$

$$\mathbf{P} = \sum_\ell \hbar\mathbf{k}_l\, a_\ell^\dagger a_\ell \tag{94}$$

Fock states

Non classicality of Fock states

Fock states are very non-classical

- A large energy
- Zero average fields and potentials since $\langle n|\, a\, |n\rangle = 0$

Can we find more intuitive field states? Yes: Coherent states.

Coherent states

Displacement operator

A unitary defined by:

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} \tag{95}$$

where α is an arbitrary complex amplitude

$$\alpha = \alpha' + i\alpha'' \tag{96}$$

$$D(\alpha)^\dagger D(\alpha) = \mathbb{1} \tag{97}$$

and

$$D(\alpha)^\dagger = D(-\alpha) \tag{98}$$

Coherent states

Displacement operator

An equivalent expression

$$D(\alpha) = e^{2i\alpha'' X_0 - 2i\alpha' P_0} \tag{99}$$

Using the Glauber relation

$$e^A e^B = e^{A+B} e^{[A,B]/2} \tag{100}$$

valid when

$$[A,[A,B]] = [B,[A,B]] = 0 \tag{101}$$

We get

$$D(\alpha) = e^{-i\alpha'\alpha''} e^{2i\alpha'' X_0} e^{-2i\alpha' P_0} \tag{102}$$

a product of displacement operators:

$$e^{-2i\alpha' P_0}\,|x\rangle = |x+\alpha'\rangle \tag{103}$$

$$e^{2i\alpha'' X_0}\,|p\rangle = |p+\alpha''\rangle \tag{104}$$

Coherent states

Combination of displacements

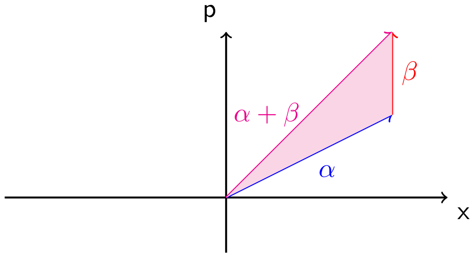
Using Glauber

$$D(\alpha)D(\beta) = e^{(\alpha\beta^* - \alpha^*\beta)/2} D(\alpha+\beta) \tag{105}$$

Note that

$$\Phi = (\alpha\beta^* - \alpha^*\beta)/2i = \frac{\alpha''\beta' - \alpha'\beta''}{2} \tag{106}$$

is the surface of the triangle with sides α and β.



Coherent states

Displacement of annihilation

Compute $D(-\alpha)aD(\alpha)$. Use Baker-Hausdorff lemma

$$e^A a e^{-A} = a + [A, a] + \frac{1}{2!}[A, [A, a]] + \dots \tag{107}$$

for $A = -\alpha a^\dagger + \alpha^* a$, with $[A, a] = \alpha$. Hence

$$D(-\alpha)aD(\alpha) = a + \alpha\mathbb{1} \tag{108}$$

Coherent states

Definition

The coherent states are defined as displaced vacuum states

$$|\alpha\rangle = D(\alpha)|0\rangle \ . \tag{109}$$

Note that $|0\rangle$ is a coherent state.
Wavefunction of a coherent state in the X_0 representation:

$$\Psi_\alpha(x) \propto e^{-(x-\alpha')^2} \tag{110}$$

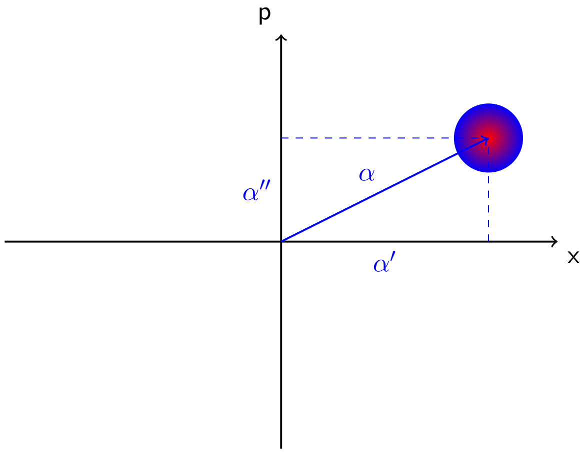
and in the P_0 representation:

$$\widetilde{\Psi}_\alpha(p) \propto e^{-(p-\alpha'')^2} \tag{111}$$

Coherent states

Pictorial representation

A displaced vacuum state



Coherent states

Properties

- Right-eigenstates of the annihilation operator

$$a|\alpha\rangle = aD(\alpha)|0\rangle = D(\alpha)D(-\alpha)aD(\alpha)|0\rangle = D(\alpha)(a+\alpha\mathbb{1})|0\rangle = \alpha|\alpha\rangle \tag{112}$$

since $a|0\rangle = 0$. Hence

$$\langle\alpha|a|\alpha\rangle = \alpha \qquad \text{and} \qquad \langle\alpha|a^\dagger|\alpha\rangle = \alpha^* \tag{113}$$

- Field operators have nonzero eigenvalues in the coherent states:

$$\begin{aligned} \langle\mathbf{E}\rangle &= i\mathcal{E}\left[\mathbf{f}(\mathbf{r})\alpha - \mathbf{f}^*(\mathbf{r})\alpha^*\right] && (114) \\ \langle\mathbf{A}\rangle &= \frac{\mathcal{E}}{\omega}\left[\mathbf{f}(\mathbf{r})\alpha + \mathbf{f}^*(\mathbf{r})\alpha^*\right] && (115) \end{aligned}$$

Coherent states

Properties

- Average photon number

$$\overline{n} = \langle\alpha|a^\dagger a|\alpha\rangle = |\alpha|^2 \tag{116}$$

- Photon number variance.
  Using $N^2 = a^\dagger a a^\dagger a = (a^\dagger)^2 a^2 + a^\dagger a$

$$\langle N^2\rangle = |\alpha|^4 + |\alpha|^2 \tag{117}$$

and

$$\Delta N^2 = |\alpha|^2 = \overline{n} \tag{118}$$

$$\frac{\Delta N}{\overline{n}} = \frac{1}{\sqrt{\overline{n}}} \tag{119}$$

Coherent states

Properties

- Expansion on the Fock state basis

$$D(\alpha) = e^{-|\alpha|^2/2}e^{\alpha a^\dagger}e^{-\alpha^* a} \tag{120}$$

with $a|0\rangle = 0$:

$$|\alpha\rangle = e^{-|\alpha|^2/2}e^{\alpha a^\dagger}|0\rangle \tag{121}$$

Expand exponential:

$$|\alpha\rangle = \sum_n c_n|n\rangle \ , \tag{122}$$

with

$$c_n = e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}} \tag{123}$$

Coherent states

Properties

- Photon number distribution

$$p_n = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!} = e^{-\overline{n}}\frac{\overline{n}^n}{n!} \tag{124}$$

For large average photon numbers $p_n \propto e^{-(n-\overline{n})^2/\overline{n}}$

- Scalar product of coherent states

$$\begin{aligned} \langle\alpha|\beta\rangle &= e^{-(|\alpha|^2+|\beta|^2)/2}\sum_{n,p}\frac{(\alpha^*)^n\beta^p}{\sqrt{n!p!}}\langle n|p\rangle \\ &= e^{-(|\alpha|^2+|\beta|^2)/2}e^{\alpha^*\beta} && (125) \end{aligned}$$

$$|\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2} \tag{126}$$

- Overcomplete basis (expansion of arbitrary state not unique)

$$\mathbb{1} = \frac{1}{\pi}\int d^2\alpha\ |\alpha\rangle\langle\alpha| \tag{127}$$

Coherent states

Properties

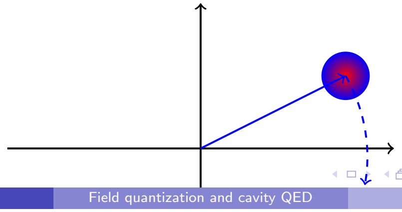
- Evolution

$$|\Psi(0)\rangle = |\alpha\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle \qquad (128)$$

$$\begin{aligned} |\Psi(t)\rangle &= e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}e^{-in\omega t}e^{-i\omega t/2}|n\rangle \\ &= e^{-i\omega t/2}\left|\alpha e^{-i\omega t}\right\rangle \qquad (129) \end{aligned}$$

Evolution of the amplitude is the same as in classical physics

$$\alpha(t) = \alpha(0)e^{-i\omega t} \qquad (130)$$



Coherent states

Production

Coherent states are produced by the interaction of a classical oscillating current (mw source, laser...)

$$\mathbf{j}(\mathbf{r},t) = \mathbf{j}_0(\mathbf{r})e^{-i\omega_0 t} \qquad (131)$$

with the field mode(s). The interaction Hamiltonian is

$$H_i = -\int_{\mathcal{V}} \mathbf{j}(\mathbf{r},t)\cdot\mathbf{A}(\mathbf{r},t)\,d^3\mathbf{r}\ . \qquad (132)$$

It is a ‘simple’ excercise to show that the evolution operator of a mode at ω for a time t is the displacement $D(\alpha)$ with

$$\alpha = -\frac{1}{\delta}\sqrt{\frac{\mathcal{V}}{2\epsilon_0\hbar\omega}}J_0\left[e^{-i\delta t}-1\right] \quad \text{where} \quad \delta = \omega_0 - \omega\ , \qquad (133)$$

reducing for $\delta = 0$ to

$$\alpha = i\sqrt{\frac{\mathcal{V}}{2\epsilon_0\hbar\omega}}J_0\,t\ . \qquad (134)$$

Phase space representations

- Classical phase space distributions $f(x,p)$ of statistical physics allowing us to compute any average by

$$\overline{o} = \int f(x,p)o(x,p)\,dxdp \qquad (135)$$

- Transpose that to a field statistical mixture defined by the density operator ρ and define a quasi-probability distribution over phase space.
- Many such phase space distributions, based on the choice of operators ordering and associated characteristic function. We examine only the two simplest.

Phase space representations

The Husimi-Q representation

Definition

$$Q^{[\rho]}(\alpha) = \frac{1}{\pi}\mathrm{Tr}\left[\rho\,|\alpha\rangle\langle\alpha|\right] = \frac{1}{\pi}\langle\alpha|\,\rho\,|\alpha\rangle = \frac{1}{\pi}\mathrm{Tr}[|0\rangle\langle 0|\,D(-\alpha)\rho D(\alpha)] \qquad (136)$$

The Q distribution is positive, bounded by $1/\pi$ and normalized ($\int d^2\alpha\, Q(\alpha) = 1$).
A few states:

- Coherent state $|\beta\rangle$

$$Q^{[|\beta\rangle\langle\beta|]}(\alpha) = \frac{1}{\pi}|\langle\alpha|\beta\rangle|^2 = \frac{1}{\pi}e^{-|\alpha-\beta|^2} \qquad (137)$$

- Fock state $|n\rangle$

$$Q^{[|n\rangle\langle n|]}(\alpha) = \frac{1}{\pi}\frac{|\alpha|^{2n}}{n!}e^{-|\alpha|^2} \qquad (138)$$

Phase space representations

The Husimi-Q representation

- Cat state

$$\left|\Psi_{\text{cat}}^{\pm}\right\rangle = \frac{1}{\sqrt{\mathcal{N}_\pm}}\left(|\beta\rangle \pm |-\beta\rangle\right) \tag{139}$$

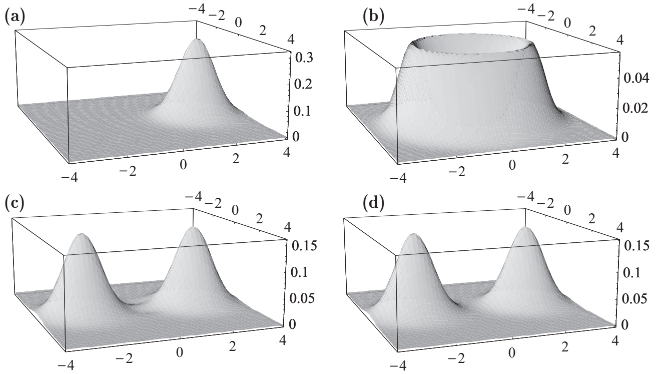
where:

$$\mathcal{N}_\pm = 2\left(1 \pm e^{-2|\beta|^2}\right) \tag{140}$$

$$Q^{[\text{cat},\pm]}(\alpha) = \frac{1}{\pi\mathcal{N}_\pm}\left[e^{-|\alpha-\beta|^2} + e^{-|\alpha+\beta|^2} \pm 2e^{-(|\alpha|^2+|\beta|^2)}\cos(2\beta\alpha'')\right] \tag{141}$$

Phase space representations

The Husimi-Q representation



(a) Coherent state $|\beta\rangle$, with $\beta = \sqrt{5}$. (b) Five-photon Fock state. (c) Schrödinger cat state, superposition of two coherent fields $|\pm\beta\rangle$, with $\beta = \sqrt{5}$. (d) Statistical mixture of the same coherent components.

Phase space representations

The Wigner function

Definition

$$W(x,p) = \frac{2}{\pi}\mathrm{Tr}[D(-\alpha)\rho D(\alpha)\mathcal{P}] \tag{142}$$

where the unitary parity operator $\mathcal{P}$ is defined by

$$\mathcal{P}\,|x\rangle = |-x\rangle \;\; ; \quad \mathcal{P}\,|p\rangle = |-p\rangle \;\; ; \quad \mathcal{P}\,|n\rangle = (-1)^n\,|n\rangle \;\; ; \quad \mathcal{P} = e^{i\pi a^\dagger a} \tag{143}$$

The modulus of its average is lower than one. Thus

$$-2/\pi \leq W(\alpha) \leq 2/\pi \tag{144}$$

Phase space representations

The Wigner function

Marginals of the Wigner distribution:

$$P(x) = \langle x|\,\rho\,|x\rangle = \int dp\, W(x,p) \tag{145}$$

and

$$P(p) = \langle p|\,\rho\,|p\rangle = \int dx\, W(x,p) \tag{146}$$

The average of any operator can be directly obtained from the Wigner function

$$\langle O\rangle = \int dxdp\, W(x,p) o_s(x,p) \tag{147}$$

where o_s is the symmetrized form of the operator O in terms of the field quadratures.

Phase space representations

The Wigner function

A few ‘classical’ states

- Coherent state

$$W^{[|\beta\rangle\langle\beta|]}(\alpha) = \frac{2}{\pi}e^{-2|\beta-\alpha|^2} \tag{148}$$

- Thermal field

$$W^{[\rho_{\rm th}]}(\alpha) = \frac{2}{\pi}\frac{1}{2n_{\rm th}+1}e^{-2|\alpha|^2/(2n_{\rm th}+1)} \tag{149}$$

- Squeezed vacuum $S(\xi)\left|0\right\rangle$ with

$$S(\xi) = e^{(\xi^* a^2-\xi a^{\dagger 2})/2} \tag{150}$$

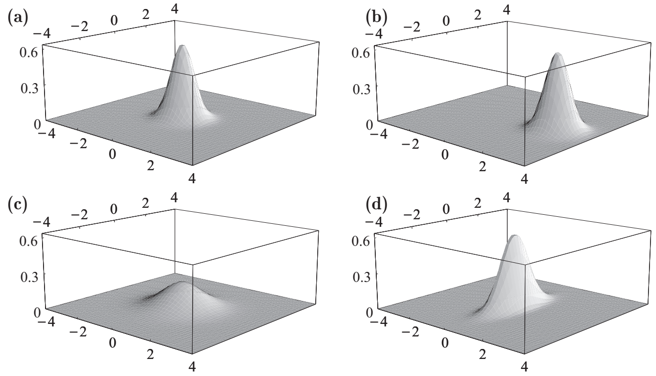
with

$$\Delta X_0 = \frac{1}{2}e^{-\xi} \quad \text{and} \quad \Delta P_0 = \frac{1}{2}e^{\xi} \tag{151}$$

$$W^{[sq,\xi]}(x,p) = \frac{2}{\pi}e^{-2\exp(2\xi)x^2}e^{-2\exp(-2\xi)p^2} \tag{152}$$

Phase space representations

The Wigner function



(a) Vacuum state. (b) Coherent state with $\beta = \sqrt{5}$. (c) Thermal field with $n_{\rm th} = 1$ photon on the average. (d) A squeezed vacuum state, with a squeezing parameter $\xi = 0.5$.

Phase space representations

The Wigner function

A few ‘non-classical’ states

- Fock state

$$W^{[|n\rangle\langle n|]}(\alpha) = \frac{2}{\pi}(-1)^n e^{-2|\alpha|^2}\mathcal{L}_n(4|\alpha|^2) \tag{153}$$

with

$$W^{[|n\rangle\langle n|]}(0) = \frac{2}{\pi}(-1)^n \tag{154}$$

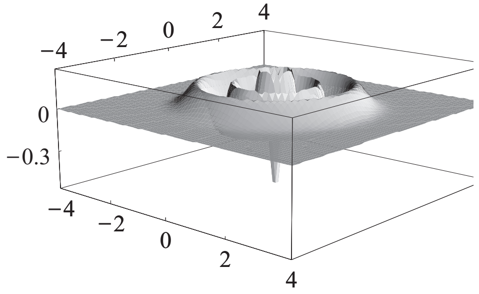
$$W^{[|1\rangle\langle 1|]}(\alpha) = -\frac{2}{\pi}(1-4|\alpha|^2)e^{-2|\alpha|^2} \tag{155}$$

- Cat state

$$\begin{aligned} W^{[\text{cat},\pm]}(\alpha) &= \frac{1}{\pi(1\pm e^{-2|\beta|^2})}\left[e^{-2|\alpha-\beta|^2}+e^{-2|\alpha+\beta|^2}\right. \\ &\pm \left. 2e^{-2|\alpha|^2}\cos(4\alpha''\beta)\right] \end{aligned} \tag{156}$$

Phase space representations

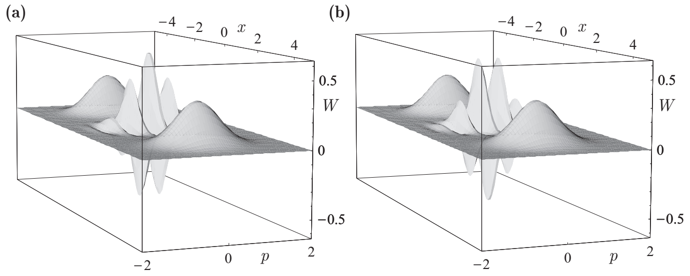
The Wigner function



Wigner function of a five-photon Fock state.

Phase space representations

The Wigner function



Wigner functions of even (a) and odd (b) 10-photon π-phase cats.

Wigner function negativities

A clear depiction of the non-classical features of a quantum state.

Field relaxation

System and environment

- Quantum system $\mathcal{S}$ (the field here, reduced to a single mode for clarity) coupled to an environment $\mathcal{E}$. Jointly in a pure state $|\Psi_{\mathcal{S}\mathcal{E}}\rangle$.
- We are interested only in $\rho_{\mathcal{S}}$, obtained by tracing the projector $|\Psi_{\mathcal{S}\mathcal{E}}\rangle \langle\Psi_{\mathcal{S}\mathcal{E}}|$ over the environment (the state of the environment is forever inaccessible).
- We seek an evolution equation for $\rho_{\mathcal{S}}$ alone.

Field relaxation

Kraus operators

- Transformation of the system's density matrix during a short time interval
$$\rho(t) \longrightarrow \rho(t+\tau) \tag{157}$$
- $\tau \gg \tau_c$, correlation time of the reservoir observables, so that there are no coherent effects in the system-reservoir interaction
- This transformation is a 'quantum map'
$$\mathcal{L}(\rho(t)) = \rho(T+\tau) \tag{158}$$

Field relaxation

Kraus operators

Mathematical properties of a proper quantum map:

- Linear operation, i.e. a super-operator in a space of dimension $N_{\mathcal{S}}^2$ ($N_{\mathcal{S}}$ system's Hilbert space dimension).
- Preserve unit trace and positivity (a density operator does not have any negative eigenvalue).
- "Completely positive". If, at a time t, $\mathcal{S}$ entangled with $\mathcal{S}'$, $\mathcal{L}$ acting on $\mathcal{S}$ alone leads to a completely positive density operator for the joint state of $\mathcal{S}$ and $\mathcal{S}'$ (not all maps are completely positive e.g. partial transpose).

Field relaxation

Kraus operators

Any completely positive map can be written as

$$\mathcal{L}(\rho) = \sum_{\mu} M_\mu \rho M_\mu^\dagger \tag{159}$$

with the normalization condition

$$\sum_{\mu} M_\mu^\dagger M_\mu = \mathbb{1} \tag{160}$$

There are at most $N_{\mathcal{S}}^2$ 'Kraus' operators M_μ, which are not uniquely defined (same map when mixing the M_μ by a linear unitary matrix V: $M'_\mu = V_{\mu\nu} M_\nu$).

Field relaxation

Kraus operators

Fit in this representation:

- Hamiltonian evolution

$$\rho(t+\tau) = U(\tau)\rho U^\dagger(\tau) \tag{161}$$

- 'unread' generalized measurement

$$\rho \longrightarrow \sum_{\mu} O_\mu \rho O_\mu^\dagger \tag{162}$$

but not a measurement whose result μ is known

$$\rho \longrightarrow \frac{O_\mu \rho O_\mu^\dagger}{\mathrm{Tr}(O_\mu \rho O_\mu^\dagger)} \tag{163}$$

(non-linear normalization term in the denominator)

Field relaxation

Lindblad equation

Kraus representation and differential representation of the map

$$\rho(t+\tau) = \sum_{\mu} M_\mu \rho M_\mu^\dagger \approx \rho(t) + \frac{d\rho}{dt}\tau \tag{164}$$

- Environment unaffected by the system: the M_μs do not depend upon time t.
- They, however, depend clearly upon the small time interval τ.
- One and only one of the M_μs is thus of the order of unity and all others must then be of order $\sqrt{\tau}$.

$$\begin{aligned} M_0 &= \mathbb{1} - iK\tau && \text{(165)} \\ M_\mu &= \sqrt{\tau} L_\mu \qquad \text{for } \mu \neq 0 && \text{(166)} \end{aligned}$$

Field relaxation

Lindblad equation

K, having no particular properties, can be split in hermitian and anti-hermitian parts:

$$K = \frac{H}{\hbar} - iJ\ , \tag{167}$$

where

$$\begin{aligned} H &= \frac{\hbar}{2}\left(K + K^\dagger\right) && \text{(168)} \\ J &= \frac{i}{2}\left(K - K^\dagger\right) && \text{(169)} \end{aligned}$$

are both hermitian.

$$M_0 = \mathbb{1} - \frac{i\tau}{\hbar} H - J\tau \tag{170}$$

Field relaxation

Lindblad equation

Thus

$$M_0\rho M_0^\dagger = \rho - \frac{i\tau}{\hbar}[H,\rho] - \tau[J,\rho]_+ \qquad (171)$$

where $[J,\rho]_+ = J\rho + \rho J$ is an anti-commutator.

$$M_0^\dagger M_0 = \mathbb{1} - 2J\tau \quad \text{and thus, by normalization since} \sum_\mu M_\mu^\dagger M_\mu = \mathbb{1} \qquad (172)$$

$$J = \frac{1}{2}\sum_{\mu\neq 0} L_\mu^\dagger L_\mu \qquad (173)$$

"Lindblad form" of the master equation

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H,\rho] + \sum_{\mu\neq 0}\left(L_\mu\rho L_\mu^\dagger - \frac{1}{2}L_\mu^\dagger L_\mu\rho - \frac{1}{2}\rho L_\mu^\dagger L_\mu\right) \qquad (174)$$

Field relaxation

Quantum jumps

Consider a single time interval τ in the simple situation where the initial state is pure $\rho(0) = |\Psi\rangle\langle\Psi|$, with no Hamiltonian evolution. Then, omitting normalization:

$$\rho(\tau) = |\Psi\rangle\langle\Psi| + \tau\sum_\mu (L_\mu|\Psi\rangle)\left(\langle\Psi|L_\mu^\dagger\right) \qquad (175)$$

- Density matrix at time τ is a statistical mixture of the initial pure state (with a large probability of order 1) and of projectors on the states $L_\mu|\Psi\rangle$.
- The L_μs are 'jump operators' which describe a random (possibly large) evolution of the system which suddenly (at the time scale of the evolution) changes under the influence of the environment.
- Intuitive picture of quantum jumps for an atom emitting a single photon

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Field relaxation

Quantum jumps

- The quantum jump operators are not defined unambiguously. Again, the same master equation can be recovered from different sets of M_μs (or L_μs) linked together by a unitary transformation matrix. Different choices correspond to different 'unravelings' of the master equation.
- In some situations, the quantum jumps have a direct physical meaning. e.g. emitting atom completely surrounded by a photo-detector array. The quantum jump then corresponds to a click of one detector. Different unravelings may then correspond to different ways of monitoring the environment, in this case to different detectors (photon counters, homodyne recievers...)
- In other situations, the quantum jumps are an abstract representation of the system+environment evolution.

Field relaxation

Quantum trajectories

- Even when the environment is not explicitly monitored, one may imagine that it is done. We then imagine we have full information about which quantum jump occurs when.
- The system is thus, at any time, in a pure state, which undergoes a stochastic trajectory in the Hilbert space, made up of continuous Hamiltonian evolutions interleaved with sudden quantum jumps.
- However, since we only imagine the information is available, we should describe the evolution of the density operator by averaging the system evolution over all possible trajectories.
- This picture leads to the Monte Carlo trajectory algorithm.

Field relaxation

Quantum Monte Carlo trajectories

- Initialize the state (randomly chosen eigenstate $|\Psi\rangle$ of ρ)
- For each time interval τ, evolve $|\Psi\rangle$ according to:
  - Compute $p_\mu = \tau \langle\Psi| L_\mu^\dagger L_\mu |\Psi\rangle$ and $p_0 = 1 - \sum_{\mu\neq 0} p_\mu$.
  - Use a (good) random number generator to decide upon the result of the measurement of $\mathcal{B}$.
  - If the result of the measurement is zero, evolve $|\Psi\rangle$ with

$$|\Psi\rangle \longrightarrow \frac{1 - iH\tau/\hbar - J\tau}{\sqrt{p_0}}|\Psi\rangle \qquad (176)$$

  - If the result of the measurement is $\mu \neq 0$, evolve $|\Psi\rangle$ by:

$$|\Psi\rangle \longrightarrow \frac{L_\mu}{\sqrt{\langle\Psi| L_\mu^\dagger l_\mu |\Psi\rangle}}|\Psi\rangle = \frac{L_\mu}{\sqrt{p_\mu/\tau}}|\Psi\rangle \qquad (177)$$

- Repeat the procedure for many trajectories
- Average the projectors $\rho(t) = \overline{|\Psi(t)\rangle\langle\Psi(t)|}$

Field relaxation

Quantum Monte Carlo trajectories

Interest of the Monte Carlo method:

- For each trajectory computes only a state vector with $N_\mathcal{S}$ dimensions i.e. $N_\mathcal{S}$ coupled differential equations, instead of $N_\mathcal{S}^2$ equations for the full density operator.
- Neeeds a statistical sample of trajectories. A few hundreds is enough to get a qualitative solution. Method more efficient than the direct integration when $N_\mathcal{S}$ is larger than a few hundreds.
- Gives a physical picture of the relaxation process (see below).

An extremely useful method, with thousands of applications.

Field relaxation

Jump operators

Only two possible jump operators at finite temperature T

- $L_- = \sqrt{\kappa_-}a$: loss of a photon in the environment (even when $T = 0$)
- $L + - = \sqrt{\kappa_+}a^\dagger$: creation of a thermal excitation

Jump rates linked to the temperature of the environment

$$\kappa_+ = \kappa_- e^{-\hbar\omega/k_b T} \qquad (178)$$

Using

$$n_{\text{th}} = \frac{1}{e^{\hbar\omega/k_b T} - 1} \qquad (179)$$

we get

$$\frac{\kappa_-}{\kappa_+} = \frac{1 + n_{\text{th}}}{n_{\text{th}}} \qquad (180)$$

and write

$$\kappa_- = \kappa(1 + n_{\text{th}}) \ ; \qquad \kappa_+ = \kappa n_{\text{th}} \qquad (181)$$

Field relaxation

Lindblad equation

$$\begin{aligned}\frac{d\rho}{dt} &= -i\omega_c\left[a^\dagger a, \rho\right] - \frac{\kappa(1 + n_{\text{th}})}{2}\left(a^\dagger a\rho + \rho a^\dagger a - 2a\rho a^\dagger\right) \\ &\quad -\frac{\kappa n_{\text{th}}}{2}\left(aa^\dagger\rho + \rho aa^\dagger - 2a^\dagger\rho a\right) \qquad (182)\end{aligned}$$

where we have discarded the vacuum energy. Note that all of the Hamiltonian part can be removed by an interaction representation (relaxation terms unchanged). For the photon number distribution:

$$\begin{aligned}\frac{dp(n)}{dt} &= \kappa(1 + n_{\text{th}})(n + 1)p(n + 1) + \kappa n_{\text{th}} np(n - 1) \\ &\quad -[\kappa(1 + n_{\text{th}})n + \kappa n_{\text{th}}(n + 1)]p(n) \qquad (183)\end{aligned}$$

Field relaxation

Thermal equilibrium

Detailed balance argument

$$\kappa(1+n_{\text{th}})np(n) = \kappa n_{\text{th}} np(n-1) \tag{184}$$

leading to:

$$\frac{p(n)}{p(n-1)} = \frac{n_{\text{th}}}{1+n_{\text{th}}} = e^{-\hbar\omega/k_b T} \tag{185}$$

The expected Maxwell equilibrium

Field relaxation

Fock states

At $T = 0$, relaxation of a Fock state

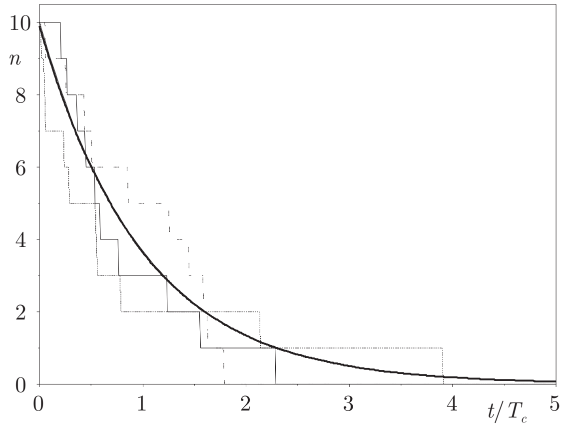
- Jump : removal of a photon
- No jump: non hermitian Hamiltonian

$$H_e = -i\hbar J = -i\hbar\kappa a^\dagger a/2 \tag{186}$$

Leaves photon number states invariant

Field relaxation

Fock states



Relaxation of a 10-photon Fock state.

Field relaxation

Coherent state

Monte Carlo trajectory

- Jump: no evolution since $|\alpha\rangle$ is an eingenstate of a
- No jumps: evolution with non hermitian hamiltonian, equivalent to a complex mode frequency

$$|\beta\rangle \rightarrow \left|\beta e^{-\kappa\tau/2}\right\rangle \tag{187}$$

A coherent state remains coherent, with an exponentially damped amplitude.

Field relaxation

Coherent state

No change of the photon number in a quantum jump ? A bayesian argument. $p(n|c)$ photon number distribution before the jump knowing that a jump occurs ('click' in the environment.) With

$$p(n,c) = p(c|n)p(n) = p(n|c)p_c \tag{188}$$

$$p(n|c) = p(n)\frac{p(c|n)}{p_c} = \frac{n}{\overline{n}}p(n) = e^{-\overline{n}}\frac{\overline{n}^{n-1}}{(n-1)!} = p(n-1) \tag{189}$$

A translated Poisson distribution with $\overline{n}+1$ photons on the average. After jump photon number unchanged. Explains why the photon number distribution is invariant in a jump. Specific property of coherent states.

Field quantization and cavity QED

Atom-field interaction

J.M. Raimond

September 23, 2019

Outline

1. Spontaneous emission in free space

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2. Photodetection

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1. Spontaneous emission in free space
2. Photodetection
3. The dressed atom model

Spontaneous emission in free space

Coupling a quantized atom to the continuum of quantized modes in free space leads to a finite lifetime for an excited level $|e\rangle$ (only the ground state is stable).
Generally many downward transitions from $|e\rangle$. All rates of these transitions add up independently (no quantum interference: final states are different). One can thus compute the rate of a single transition between $|e\rangle$ and a lower non-degenerate state $|g\rangle$.

- Write the atom-field hamiltonian
- Solve the Schrödinger equation (Wigner Weisskopf approach)
- Compute emission rate and (possible) energy shifts

Spontaneous emission

The atom-field Hamiltonian

Hydrogen atom in a classical radiation field [potentials $\mathbf{A}(\mathbf{r}, t)$ and $\mathbf{V}(\mathbf{r}, t)$]

$$H = \frac{1}{2m}\left(\mathbf{P} - q\mathbf{A}(\mathbf{R}, t)\right)^2 + qU(\mathbf{R}) + qV(\mathbf{R}, t) \tag{1}$$

Two approximations

- Linear approximation : neglect second order tems in $\mathbf{A}$ in the Hamiltonian (legitimate if electric field $\ll$ atomic units).
- Dipole approximation : neglect atomic size with respect to field characteristic wavelength

Two equivalent forms of the interaction Hamiltonian, with $H = H_0 + H_{ap}$

$$H_{ap} = -\frac{q}{m}\mathbf{P}\cdot\mathbf{A}(0) \tag{2}$$

$$H_{de} = -\mathbf{D}\cdot\mathbf{E}(0) \tag{3}$$

where $\mathbf{D} = q\mathbf{R}$. Assume these forms are OK with quantized $\mathbf{A}$ and $\mathbf{E}$.

Spontaneous emission in free space

Wigner Weisskopf approach

Atom-field state at time t:

$$|\Psi(t)\rangle = c_0(t)\,|e, 0\rangle + \sum_\ell c_\ell(t)\,|g, 1_\ell\rangle \tag{4}$$

Schrödinger equation:

$$i\hbar\frac{dc_0}{dt} = \hbar\omega_{eg}c_0 + \sum_\ell V_\ell c_\ell \tag{5}$$

$$i\hbar\frac{dc_\ell}{dt} = \hbar\omega_\ell c_\ell + V_\ell^* c_0 \tag{6}$$

$$V_\ell = -\langle e, 0|\,\mathbf{D}\cdot\mathbf{E}\,|g, 1_\ell\rangle \tag{7}$$

Interaction representation:

$$b_\ell = c_\ell e^{i\omega_\ell t} \qquad i\hbar\frac{db_\ell}{dt} = e^{i\omega_\ell t}V_\ell^* c_0 \tag{8}$$

Wigner-Weisskopf

Formal integration

$$b_\ell(t) = \frac{V_\ell^*}{i\hbar}\int_0^t c_0(t')e^{i\omega_\ell t'}\,dt' \tag{9}$$

or

$$c_\ell(t) = \frac{V_\ell^*}{i\hbar}\int_0^t c_0(t')e^{i\omega_\ell(t'-t)}\,dt' \tag{10}$$

Setting

$$c_0 = e^{-i\omega_{eg}t}\alpha_0(t) \tag{11}$$

We get

$$\frac{d\alpha_0}{dt} = -\sum_\ell \frac{|V_\ell|^2}{\hbar^2}e^{i\omega_{eg}t}\int_0^t e^{i\omega_\ell(t'-t)}e^{-i\omega_{eg}t'}\alpha_0\,dt' \tag{12}$$

Wigner-Weisskopf

Changing for the variable $\tau = t - t'$, we get

$$\frac{d\alpha_0}{dt} = -\int_0^t \mathcal{N}(\tau)\alpha_0(t-\tau)\,d\tau \tag{13}$$

where the integral kernel $\mathcal{N}$ is:

$$\mathcal{N}(\tau) = \frac{1}{\hbar^2}\sum_\ell |V_\ell|^2 e^{i(\omega_{eg}-\omega_\ell)\tau} \tag{14}$$

Wigner-Weisskopf

Let us calculate $V_\ell = -\langle e,0|\,\mathbf{D}\cdot\mathbf{E}\,|g,1_\ell\rangle$.
Without loss of generality $\mathbf{D} = qZ\mathbf{u}_z$.
Using $\mathbf{E}(0) = i\sum_\ell \sqrt{\frac{\hbar\omega_\ell}{2\epsilon_0\mathcal{V}}}\,a_\ell\boldsymbol{\epsilon}_\ell + \text{h.c.}$ we get

$$|V_\ell|^2 = \frac{\hbar\omega_\ell}{2\epsilon_0\mathcal{V}}|d|^2|\mathbf{u}_z\cdot\boldsymbol{\epsilon}_\ell|^2 \tag{15}$$

with $d = \langle e|\,qZ\,|g\rangle$.
Hence,

$$\mathcal{N}(\tau) = \frac{|d|^2}{\hbar^2}\left[\sum_\ell |\mathbf{u}_z\cdot\boldsymbol{\epsilon}_\ell|^2\frac{\hbar\omega_\ell}{2\epsilon_0\mathcal{V}}e^{-i\omega_\ell\tau}\right]e^{i\omega_{eg}\tau} \tag{16}$$

In the square brackets, a sum of very many oscillations with a large span of frequencies. In a time of the order of $1/\omega_{eg}$, $\mathcal{N}$ practically vanishes

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Wigner-Weisskopf

Thus:

$$\int_0^t \mathcal{N}(\tau)\alpha(t-\tau)\,d\tau \approx \alpha_0(t)\int_0^\infty \mathcal{N}(\tau)\,d\tau = \left(\frac{\Gamma}{2}+i\Delta\right)\alpha_0(t) \tag{17}$$

$$\frac{d\alpha_0}{dt} = -\left(\frac{\Gamma}{2}+i\Delta\right)\alpha_0 \tag{18}$$

- Γ spontaneous emission rate
- Δ level shift

Wigner-Weisskopf

Final solution

$$c_0(t) = e^{-\Gamma t/2}e^{-i\omega_0 t}e^{-i\Delta t} \tag{19}$$

$$c_\ell(t) = \frac{V_\ell}{i\hbar}\frac{1-e^{-\Gamma t/2}e^{i(\omega_\ell-\omega_0-\Delta)t}}{(\Gamma/2)-i(\omega_\ell-\omega_0-\Delta)} \tag{20}$$

$$|c_\ell(\infty)|^2 = \frac{|V_\ell|^2}{\hbar^2}\frac{1}{(\Gamma^2/4)+(\omega_\ell-\omega_0-\Delta)^2} \tag{21}$$

a lorentzian profile for the spontaneous emission line.

Wigner-Weisskopf

Decay and shifts

Explicit integration of the kernel:

$$\left(\frac{\Gamma}{2}+i\Delta\right)=\frac{|d|^2}{\hbar^2}\sum_\ell(\mathbf{u}_z\cdot\boldsymbol{\epsilon}_\ell)^2\frac{\hbar\omega_\ell}{2\epsilon_0\mathcal{V}}\int_0^\infty e^{i(\omega_{eg}-\omega_\ell)\tau}\,d\tau \tag{22}$$

Using

$$\int_0^\infty e^{i\omega t}\,dt=\pi\delta(t)+i\mathcal{P}\mathcal{P}\frac{1}{\omega} \tag{23}$$

we get for the real part:

$$\Gamma=\frac{2\pi|d|^2}{\hbar^2}\sum_\ell(\mathbf{u}_z\cdot\boldsymbol{\epsilon}_\ell)^2\frac{\hbar\omega_\ell}{2\epsilon_0\mathcal{V}}\delta(\omega_{eg}-\omega_\ell) \tag{24}$$

Wigner-Weisskopf

Spontaneous emission rate

Replace sum over modes by an integral:

$$\sum_\ell\longrightarrow\sum_{\boldsymbol{\epsilon}_\ell}\int d\Omega\int d\nu_\ell\frac{\rho(\nu)}{4\pi}\ , \tag{25}$$

where $d\Omega$ is a fraction of the solid angle centered on the emission direction $\mathbf{u_k}$, $\rho(\nu)$ is the free space mode density and the finite sum extends on the two polarizations for each propagation direction.

Wigner-Weisskopf

Plane-wave basis: counting the modes

N_ν the total number of modes $k<2\pi\nu/c$. Number of modes between ν and $\nu+d\nu$: $\rho_\nu\,d\nu$

$$\rho_\nu=\frac{dN_\nu}{d\nu} \tag{26}$$

Counting the modes with a frequency lower than ν amounts to counting twice (two polarizations) the number of points with integer coordinates in a sphere of radius $2\pi\nu/c$:

$$N_\nu=2\frac{\frac{4\pi}{3}\left(\frac{2\pi\nu}{c}\right)^3}{\frac{8\pi^3}{\mathcal{V}}}=\frac{8\pi}{3}\frac{\nu^3}{c^3}\mathcal{V}\ . \tag{27}$$

Hence

$$\rho_\nu=\frac{8\pi}{c^3}\mathcal{V}\nu^2 \tag{28}$$

Wigner-Weisskopf

Spontaneous emission rate

With $\rho(\nu)=8\pi\mathcal{V}\nu^2/2c^3$, using $\delta(\omega_{eg}-\omega_\ell)=\delta(\nu_{eg}-\nu_\ell)/2\pi$ and performing the trivial integration on ν_ℓ, we get

$$\Gamma=\frac{d^2\omega_{eg}^3}{8\pi\hbar\epsilon_0c^3}\int\sum_{\boldsymbol{\epsilon}_\ell}(\mathbf{u}_z\cdot\boldsymbol{\epsilon}_\ell)^2d\Omega \tag{29}$$

Wigner-Weisskopf

Spontaneous emission rate

Expand $\mathbf{u}_z$ on the basis of $\mathbf{u_k}$ (propagation direction) and two orthogonal linear polarizations ϵ_1 and ϵ_2:

$$(\mathbf{u}_z \cdot \epsilon_1^*)^2 + (\mathbf{u}_z \cdot \epsilon_2^*)^2 = 1 - (\mathbf{u}_z \cdot \mathbf{u_k})^2 = 1 - \cos^2\theta = \sin^2\theta \qquad (30)$$

Integration over solid angle:

$$\int \sum_{\epsilon_\ell} (\mathbf{u}_z \cdot \epsilon_\ell)^2 d\Omega = \int_0^{2\pi} \int_0^{\pi} \sin^3\theta \, d\theta d\phi = \frac{8\pi}{3} \qquad (31)$$

and finally

$$\Gamma = \frac{|d|^2 \omega_{eg}^3}{3\pi\omega_0 \hbar c^3} \qquad (32)$$

Wigner-Weisskopf

Energy shifts

Level shift

A severe problem

Δ is divergent

A (not so simple) solution

Renormalization

However, fortunately, it is not needed for most atomic physics situation. Include the shifts due to all other modes than the mode of interest in the energy levels, and compute interaction of this ‘renormalized’ atom with the only mode of interest.

Photodetector model

A simple single-system photodetector. A ground state $|g\rangle$ and a continuum of excited states $|e_i\rangle$. Transition to excited state is a click.
Detector Hamiltonian

$$H_d = \sum_i \hbar\omega_i \, |e_i\rangle \langle e_i| \qquad (33)$$

Detector-field interaction $-\mathbf{D} \cdot \mathbf{E}$ with

$$\mathbf{D} = \sum_i d_i(\epsilon_i \, |g\rangle \langle e_i| + \epsilon_i^* \, |e_i\rangle \langle g|) \qquad (34)$$

Hence, within irrelevant factors

$$H_i = \sum_i \kappa_i \, |e_i\rangle \langle g| \, E^+ + \text{h.c.} \qquad (35)$$

Photodetector model

Interaction representation $a_\ell \rightarrow a_\ell \exp(-i\omega_\ell t)$ $E \rightarrow E(t)$,
$|e_i\rangle \langle g| \rightarrow \exp(i\omega_i t) \, |e_i\rangle \langle g|$

$$\widetilde{H}_i = \sum_i \kappa_i e^{i\omega_i t} \, |e_i\rangle \langle g| \, E^+(t) + \text{h.c.} \qquad (36)$$

Initial condition

$$|\Psi(0)\rangle = |g\rangle \otimes |\Psi_f\rangle \qquad (37)$$

State at time t

$$|\Psi(t)\rangle = |g\rangle \otimes |\Psi_f\rangle + \frac{1}{i\hbar} \int_0^t \widetilde{H}_i(t') \, |\Psi(t')\rangle \, dt' \qquad (38)$$

First-order perturbative solution by replacing in the r.h.s. $|\Psi(t')\rangle$ by $|\Psi(0)\rangle = |g\rangle \otimes |\Psi_f\rangle$.

Photodetector model

Noting that, in $\widetilde{H}_i$, $|g\rangle \langle e_i| E^-$ gives zero on the initial state

$$|\Psi(t)\rangle = |g\rangle \otimes |\Psi_f\rangle + \frac{1}{i\hbar}\sum_i \kappa_i \left[\int_0^t dt'\, e^{i\omega_i t'} E^+(t')\,|\Psi_f\rangle\right] \otimes |e_i\rangle \tag{39}$$

Probability for having a count at time t

$$p_e = \sum_i |\langle e_i\,|\Psi\rangle|^2 = \sum_i \langle\Psi\,|e_i\rangle\,\langle e_i\,|\Psi\rangle \tag{40}$$

$$p_e = \frac{1}{\hbar^2}\sum_i |\kappa_i|^2 \int_0^t dt' \int_0^{t'} dt''\, e^{i\omega_i(t'-t'')}\,\langle\Psi_f|\, E^-(t'')E^+(t')\,|\Psi_f\rangle \tag{41}$$

Photodetector model

For a high density of final states

$$\sum_i \longrightarrow \int d\omega\rho(\omega) \tag{42}$$

$$\int d\omega e^{i\omega(t'-t'')} = \pi\delta(t'-t'') \tag{43}$$

Hence

$$p_e(t) \propto \int_0^t dt'\ \langle\Psi_f|\, E^-(t')E^+(t')\,|\Psi_f\rangle \tag{44}$$

With a large set of photo-detecting systems the ‘photocurrent’ is proportional to

$$I(t) = \langle\Psi_f|\, E^-(t)E^+(t)\,|\Psi_f\rangle \tag{45}$$

Dressed atom

A frequent situation in quantum optics: a two level atom coupled to a single mode of the radiation field. Coherent coupling larger than dissipative process.

- An atom in an intense laser field
- Cavity quantum electrodynamics

An ideal situation

A two-level atom coupled to a single field mode. Or a spin 1/2 coupled to a harmonic oscillator. The simplest non-trivial quantum system.

Almost ideally implemented in Cavity Quantum Electrodynamics, circuit QED but also in ion traps. Much more on that soon.
Fruitful to treat atom and mode as a single quantum system: the Dressed atom (Cohen-Tannoudji)

A two-level system

We consider the case of a single radiation mode at frequency ω_0 resonant or nearly resonant on the transition between the two levels $|g\rangle$ (lower, possibly ground level) and $|e\rangle$ i.e.

$$\omega_0 \approx \omega_{eg}$$

All other levels can be neglected.

Free atom

Two states $|e\rangle$ and $|g\rangle$ or $|+\rangle$ and $|-\rangle$ or $|0\rangle$ and $|1\rangle$ in quantum information science. Equivalent to a spin-1/2 system.
Operator basis set: Pauli operators

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{46}$$

$$[\sigma_x, \sigma_y] = 2i\sigma_z \tag{47}$$

Spin lowering and raising operators

$$\sigma_+ = |+\rangle\langle -| = \frac{\sigma_x + i\sigma_y}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{48}$$

$$\sigma_- = |-\rangle\langle +| = \sigma_+^\dagger = \frac{\sigma_x - i\sigma_y}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{49}$$

$$[\sigma_z, \sigma_\pm] = \pm 2\sigma_\pm \tag{50}$$

Free atom

Most general observable $\sigma_{\mathbf{u}}$ with $\mathbf{u} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$$\sigma_{\mathbf{u}} = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix} \tag{51}$$

Eigenvectors

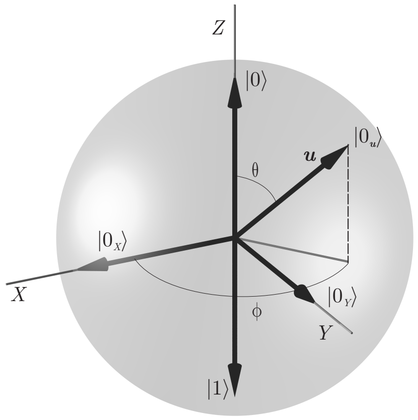
$$|+_{\mathbf{u}}\rangle = |0_{\mathbf{u}}\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}e^{i\phi}|-\rangle \tag{52}$$

$$|-_{\mathbf{u}}\rangle = |1_{\mathbf{u}}\rangle = -\sin\frac{\theta}{2}e^{-i\phi}|+\rangle + \cos\frac{\theta}{2}|-\rangle \tag{53}$$

Span the whole Hilbert space for all values of θ and ϕ.

Free atom

Bloch sphere



Free atom

Going from one state to another by a rotation on the Bloch sphere by an angle θ around the axis defined by $\mathbf{v}$

$$R_{\mathbf{v}}(\theta) = e^{-i(\theta/2)\sigma_{\mathbf{v}}} = \cos\frac{\theta}{2}\mathbb{1} - i\sin\frac{\theta}{2}\sigma_{\mathbf{v}} \tag{54}$$

e.g. angle θ around $\mathbf{u}_z$

$$R_z(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \tag{55}$$

with $R_z(\pi/2)|+_x\rangle = |+_y\rangle$ and $R_{\mathbf{v}}(2\pi) = -\mathbb{1}$

Free atom

- Hamiltonian:
$$H_a = \frac{\hbar\omega_{eg}}{2}\sigma_z \tag{56}$$
Generates a rotation of the Bloch vector at angular frequency ω_{eg} around Oz (Larmor precession in the NMR context).
- Dipole operator:
$$\mathbf{D} = d(\boldsymbol{\epsilon}_a\sigma_- + \boldsymbol{\epsilon}_a^*\sigma_+) \tag{57}$$
where $\boldsymbol{\epsilon}_a$ describes the polarization of the atomic transition.

Atomic relaxation

Spontaneous emission can be treated as a perturbation (weak effect compared to atom-field interaction). Described in terms of quantum jumps and Lindblad equations. We assume a zero-temperature environment (valid for optical transitions).
A single jump operator (describing photon emission in a downwards transition)

$$L = \sqrt{\Gamma}\sigma_- \tag{58}$$

with $\Gamma = 1/T_1$ ('longitudinal relaxation time'). Lindblad equation

$$\frac{d\rho}{dt} = \Gamma\left(\sigma_-\rho\sigma_+ - \frac{1}{2}\sigma_+\sigma_-\rho - \frac{1}{2}\rho\sigma_+\sigma_-\right) \tag{59}$$

Atomic relaxation

With

$$\rho = \begin{pmatrix} \rho_{ee} & \rho_{eg} \\ \rho_{ge} & \rho_{gg} \end{pmatrix} \tag{60}$$

the Lindblad equation can be rewritten as:

$$\frac{d\rho_{ee}}{dt} = -\Gamma\rho_{ee} \tag{61}$$

$$\frac{d\rho_{eg}}{dt} = -\frac{\Gamma}{2}\rho_{eg} \tag{62}$$

- Relaxation of excited state population with a rate Γ.
- Relaxation of coherence with a rate $\Gamma/2$ (compatible with $\rho_{eg} \leq \sqrt{\rho_{ee}\rho_{gg}}$)

Atomic relaxation

Case of an initial superposition state $|\Psi_0\rangle = (1/\sqrt{2})(|e\rangle + |g\rangle)$. Analysis in terms of the Monte Carlo trajectories.

- No jump evolution. With $|\Psi(t)\rangle = c_e|e\rangle + c_g|g\rangle$, we use the effective Hamiltonian
$$H = -i\hbar J = -\frac{i\hbar}{2}\Gamma\sigma_+\sigma_- = -\frac{i\hbar}{2}\Gamma|e\rangle\langle e| \tag{63}$$
$$i\hbar\frac{dc_e}{dt} = -\frac{i\hbar}{2}\Gamma c_e \qquad c_e(t) = c_e(0)e^{-\Gamma t/2} \qquad \frac{dc_g}{dt} = 0 \tag{64}$$
$$|\Psi(t)\rangle = \frac{1}{|c_e(0)|^2e^{-\Gamma T} + |c_g(0)|^2}\left(c_e(0)e^{-\Gamma t/2}|e\rangle + c_g(0)|g\rangle\right) \tag{65}$$
A negative detection (no photon emitted) changes the system's state.
- Jump: state becomes $|g\rangle$. No further evolution.

The single mode

Field normalization

A slightly different choice of normalization for the spatial mode function. In free space: the volume is that of the fictitious quantization box. For an actual mode, or a cavity mode with a known geometry, less ambiguous to set

$$f(\mathbf{r}) = 1 \tag{66}$$

at the field maximum. This defines the mode volume as

$$\mathcal{V} = \int |f(\mathbf{r})|^2 \, d^3\mathbf{r} \tag{67}$$

Since we will soon refer explicitly to a cavity mode with angular frequency ω_c, use an index c for field operators.

Atom-field interaction

A single mode: get rid of vacuum energy and use the mode Hamiltonian

$$H'_c = \hbar\omega_c N \tag{68}$$

The exact atom-mode Hamiltonian is then

$$H = H_a + H'_c + H_{ac} \ , \tag{69}$$

with

$$H_{ac} = -\mathbf{D} \cdot \mathbf{E}_c = -d(\epsilon_a\sigma_- + \epsilon_a^*\sigma_+) \cdot i\mathcal{E}_0(\epsilon_c a f(\mathbf{r}) - \epsilon_c^* a^\dagger f^*(\mathbf{r})) \tag{70}$$

We then perform the standard Rotating wave approximation, neglecting the action of the far off-resonant terms when the frequency of the atomic transition is close to that of the mode. We thus drop terms in $a^\dagger\sigma_+$ and $a\sigma_-$.

Atom-field interaction

For the sake of simplicity, we assume that the atom is sitting at the mode center (the field maximum) corresponding to $f(\mathbf{r}) = 1$. Then,

$$H_{ac} = -i\hbar\frac{\Omega_0}{2}[a\sigma_+ - a^\dagger\sigma_-] \tag{71}$$

with

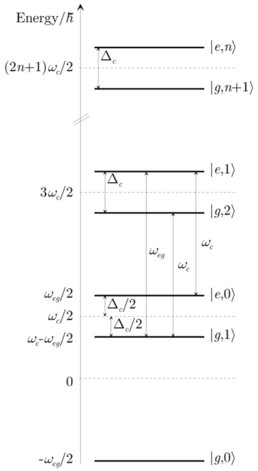
$$\Omega_0 = 2\frac{d\mathcal{E}_0\epsilon_a^* \cdot \epsilon_c}{\hbar} \ . \tag{72}$$

Ω_o is called the “vacuum Rabi frequency” and we will (without loss of generality) assume it to be real.
H is the Jaynes-Cummings Hamiltonian (1963).
Let us first examine the uncoupled quantum states.

Uncoupled states

The eigenstates of $H_a + H'_c$ are the uncoupled states $|e, n\rangle$ and $|g, n\rangle$. Assuming $\Delta_c = \omega_{eg} - \omega_0 \ll \omega_{eg}$, all levels $|e, n\rangle$ and $|g, n+1\rangle$ are nearly degenerate (energy separation $\hbar\Delta_c$). The ground state $|g, 0\rangle$ is an isolated one, obviously impervious to the atom-mode interaction.

Uncoupled states



Dressed states

Diagonalizing the Hamiltonian

H_{ac} only couples the nearly degenerate states $|e,n\rangle$ and $|g,n+1\rangle$. The diagonalization of the total Hamiltonian can be performed as separate diagonalizations of 2×2 Hamiltonians.
Use an interation representation setting the energy origin at the center of the manifold. Final matrix form of the total Hamiltonian H in the $\{|e,n\rangle$, $|g,n+1\rangle\}$ basis

$$\widetilde{H}=\frac{\hbar}{2}\begin{pmatrix}\Delta_c & -i\Omega_n\\ i\Omega_n & \Delta_c\end{pmatrix} \tag{73}$$

where

$$\Omega_n=\Omega_0\sqrt{n+1} \tag{74}$$

is the n-photon Rabi frequency

Dressed states

The dressed states, eigenstates of the full Hamiltonian have the energies

$$E_n^{\pm}=\pm\frac{\hbar}{2}\sqrt{\Delta_c^2+\Omega_n^2} \tag{75}$$

They are

$$|\pm,n\rangle=\cos\theta_n^{\pm}\,|e,n\rangle+i\sin\theta_n^{\pm}\,|g,n+1\rangle \tag{76}$$

with

$$\tan\theta_n^{\pm}=\pm\frac{\sqrt{\Delta_c^2+\Omega_n^2}\mp\Delta_c}{\Omega_n} \tag{77}$$

The expressions are generally complex. Simple cases at exact resonance ($\Delta_c=0$) and in the dispersive, nonresonant regime ($\Delta_c\gg\Omega_n$)

Dressed states

Resonant regime

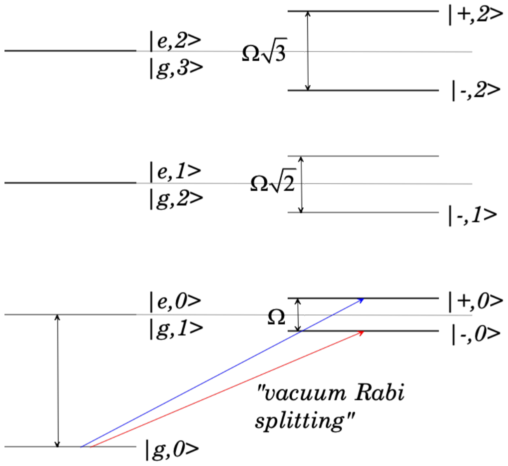
At resonance

$$|\pm,n\rangle=\frac{1}{\sqrt{2}}\left[|e,n\rangle\pm i\,|g,n+1\rangle\right] \tag{78}$$

Dressed states are equal-weight superpositions of the uncoupled states.
The splitting of the dressed manifold is $\hbar\Omega_n$.
System initially prepared in $|e,n\rangle$: Rabi oscillation between the two uncoupled states at Ω_n.

Dressed states

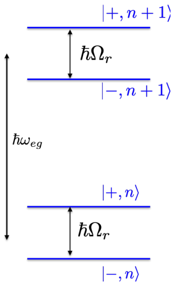
Resonant regime



Dressed states

Classical Rabi oscillation

Case of a large coherent field $\overline{n} \gg 1$. We can neglect the variation of Ω_n over the range of populated n values. The dressed states multiplicities have then a constant splitting $\Omega_r = \Omega_0\sqrt{\overline{n}}$.

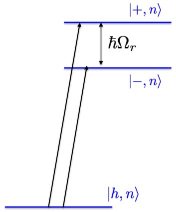


The field state is not affected by atomic emission/absorption and we recover the classical Rabi oscillation at Ω_r as a quantum beat between dressed states.

Dressed states

Autler-Townes splitting

Probe the dressed level structure by inducing the $|h\rangle \rightarrow |e\rangle$ transition (h is a level uncoupled to the single mode).

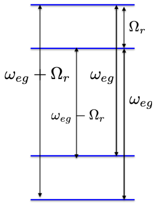


A doublet of lines separated by Ω_r.

Dressed states

Mollow triplet

Add atomic relaxation (spontaneous emission). Both dressed states have an e character and can thus both decay towards dressed states in the immediately lower manifold (the uncoupled states decay is from $|e, n\rangle$ to $|g, n\rangle$).



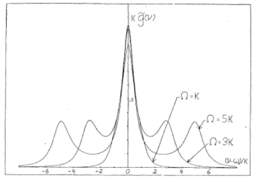


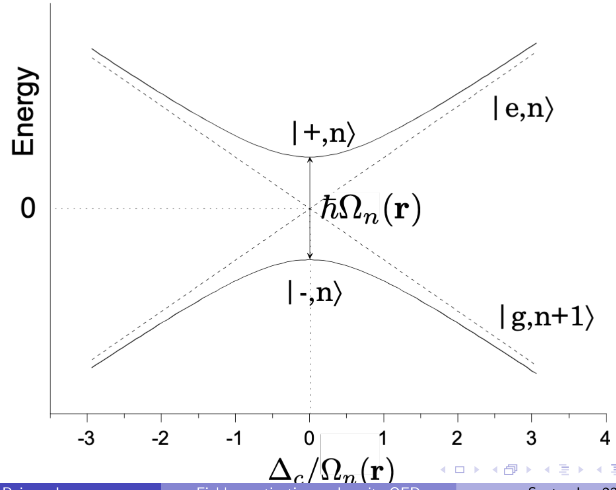
FIG. 1. Spectral density $\tilde{g}(\nu)$ for a two-level atom driven exactly on resonance.

The emission spectrum is a triplet of lines (Mollow Phys. Rev. 188 1969 (1969))

Dressed states

Nonesonant regime

Position of the dressed states as a function of the detuning.



Dressed states

Dispersive regime

Large detuning limit: $\Delta_c \gg \Omega_n$. $|+,n\rangle \rightarrow |e,n\rangle$ and $|-,n\rangle \rightarrow -i\,|g,n+1\rangle$ and

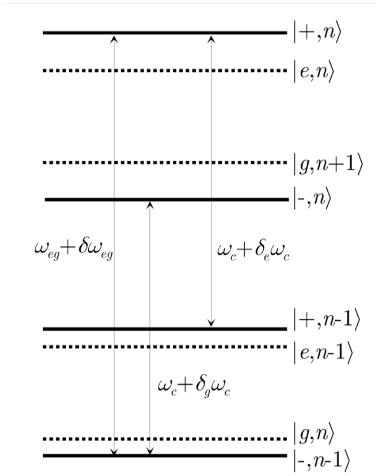
$$E_n^{\pm} = \pm\hbar\left(\frac{\Delta_c}{2} + \frac{\Omega_n^2}{4\Delta_c}\right) \tag{79}$$

In other words, $|e,n\rangle$ is shifted by $\hbar s(n+1)$ and $|g,n+1\rangle$ is shifted by $-\hbar sn$, where

$$s = \frac{\Omega_n^2}{4\Delta_c} \tag{80}$$

Dressed states

Dispersive regime



Dressed states

Dispersive regime

Interpretation: light and index shifts.

- Lamb and Light shifts: the atomic frequency is modified by

$$\delta\omega_{eg} = s(2n+1)$$

- Atomic index of refraction:

$$\delta_e\omega_c = s$$

and

$$\delta_g\omega_c = -s$$

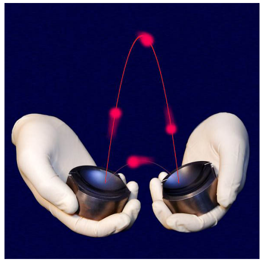
Dressed states

Cavity QED

Let us now discuss all that in more details in the Cavity Quantum Electrodynamics context.

Field quantization:
cavity quantum electrodynamics

J.M. Raimond
Sorbonne Université
LKB, Collège de France, ENS, CNRS, SU



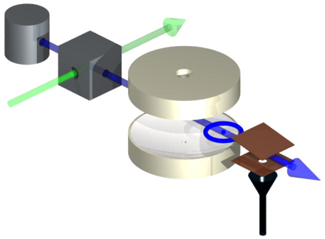
RYSQ ANR QuSCo erc TRENSCRYBE

Outline of this lecture

- Introduction
- Tools of cavity QED
- Resonant interaction
- Dispersive interaction
- Perspectives

Cavity Quantum Electrodynamics

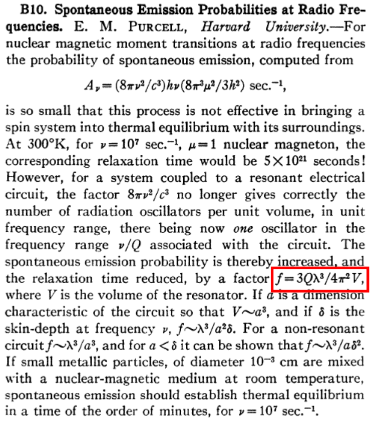
- A spin and a spring



  - Realizes the simplest matter-field system: a single atom coherently coupled to a few photons in a single mode of the radiation field.
  - Direct illustrations of quantum postulates

A history of CQED: the origin

- Purcell 1946
  - spontaneous emission rate modification for a spin in a resonant circuit
  - Definition of the 'Purcell factor'
  - Brief but seminal
- Kleppner 81
  - Inhibition of spontaneous emission



B10. Spontaneous Emission Probabilities at Radio Frequencies. E. M. Purcell, *Harvard University*.—For nuclear magnetic moment transitions at radio frequencies the probability of spontaneous emission, computed from

$$A_\nu = (8\pi\nu^2/c^3)h\nu(8\pi^3\mu^2/3h^2)\ \text{sec.}^{-1},$$

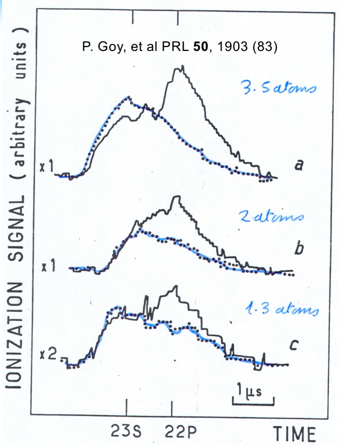
is so small that this process is not effective in bringing a spin system into thermal equilibrium with its surroundings. At 300°K, for $\nu = 10^7$ sec.$^{-1}$, $\mu = 1$ nuclear magneton, the corresponding relaxation time would be 5×10^{21} seconds! However, for a system coupled to a resonant electrical circuit, the factor $8\pi\nu^2/c^3$ no longer gives correctly the number of radiation oscillators per unit volume, in unit frequency range, there being now *one* oscillator in the frequency range ν/Q associated with the circuit. The spontaneous emission probability is thereby increased, and the relaxation time reduced, by a factor $\boxed{f = 3Q\lambda^3/4\pi^2 V,}$ where V is the volume of the resonator. If a is a dimension characteristic of the circuit so that $V \sim a^3$, and if δ is the skin-depth at frequency ν, $f \sim \lambda^3/a^2\delta$. For a non-resonant circuit $f \sim \lambda^3/a^3$, and for $a < \delta$ it can be shown that $f \sim \lambda^3/a\delta^2$. If small metallic particles, of diameter 10^{-3} cm are mixed with a nuclear-magnetic medium at room temperature, spontaneous emission should establish thermal equilibrium in a time of the order of minutes, for $\nu = 10^7$ sec.$^{-1}$.

The two regimes of cavity QED

- Weak coupling regime
  - Atom-field coupling small compared to dissipation
    - No qualitative modifications of the atomic radiative properties
      - Modification of the spontaneous emission rate
      - Modification of the atomic energies
- Strong coupling regime
  - Atom-cavity interaction overwhelms dissipative processes
    - The simplest matter-field coupling situation
      - Radical modification of the atomic radiative properties
      - Creates and manipulates atom/field entangled state

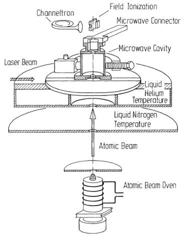
First experiment on weak coupling

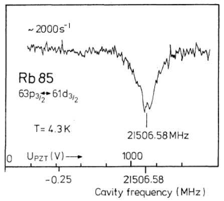
- Spontaneous emission enhancement
  - Superconducting FP cavity
    - Q α 10^6
    - 340 GHz transition
  - Acceleration x 530
  - First experimental evidence of Purcell effect
- Spontaneous emission inhibition
  - Gabrielse and Dehmelt (85)
  - Hulet, Hilfer and Kleppner (85)
- Spontaneous emission can be altered at will by imposing limiting conditions to the field


P. Goy, et al PRL **50**, 1903 (83)

First experiment on strong coupling: the micromaser

- H. Walther and D. Meschede, 85
  - Cumulative emissions in the cavity in the strong coupling regime





  - A maser with less than one atom at a time in the cavity
  - A new type of quantum oscillator. Role of quantum fluctuations
  - Strong coupling regime
    - Single-Atom-cavity coupling overwhelms dissipation

The four time scales of CQED

- Atomic levels lifetime

$$T_{at} = 1/\Gamma$$

- Cavity lifetime

$$T_c = 1/\kappa$$

- Atom-cavity coupling

$$\Omega_0 = 2g = 1/T_{res}$$

- Atom-cavity interaction time

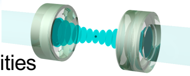
$$T_{\text{int}}$$

- Strong coupling conditions

$$T_{\text{int}}\Omega_0 \approx 1; \quad T_{res}, T_{\text{int}} << T_{at}, T_c$$

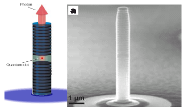
The four flavours of modern CQED

- Optical CQED
  - Ordinary atomic transitions and high finesse FP cavities



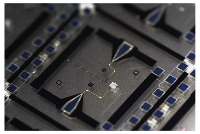
$g \approx 50\ \text{MHz};\ \kappa \approx 100\ \text{kHz};\ \Gamma \approx 10\ \text{MHz};\ T_{\text{int}} \approx 1\text{s}$

- Solid-state CQED
  - Quantum dots coupled to bragg mirrors/PBG



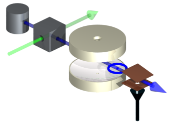
$g \approx 10\ \text{GHz};\ \kappa \approx 1\ \text{GHz};\ \Gamma \approx\ 1\text{GHz};\ T_{\text{int}} = \infty$

- Circuit QED
  - Solid-state qubits and stripline cavities



$g \simeq 100MHz; \Gamma \ll \kappa \simeq 1MHz; T_{int} = \infty$

- Microwave CQED
  - (Circular) Rydberg atoms and superconducting cavities



$g \approx 10\ \text{kHz};\ \kappa \approx 1\ \text{Hz};\ \Gamma \approx 30\ \text{Hz};\ T_{\text{int}} \approx 100\ \mu s$

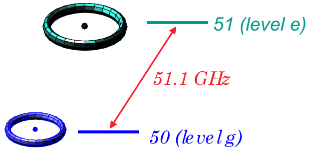
These lectures

- Focus on microwave QED
  - Paradigmatic example of CQED
  - Some (hopefully) interesting experiments

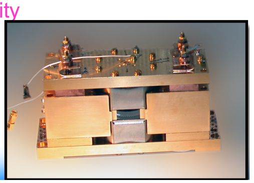
Outline of this lecture

- Introduction
- Tools of cavity QED
- Resonant interaction
- Dispersive interaction
- Perspectives

Microwave CQED experiments at ENS (1983-…)



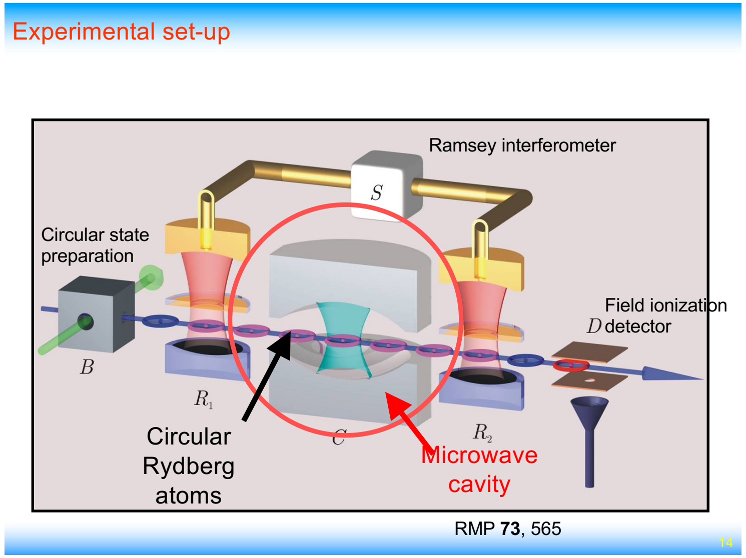
- Two ideal tools
  - Circular Rydberg atoms
    - Ideal two-level atoms
      - Long lifetime (30ms)
      - Microwave two-level transition
      - Stark tuning
      - Huge dipole matrix element
      - Selective and sensitive detection
        - » Field ionization
      - Poissonian atom number in a laser excited sample
        - » No deterministic one atom preparation
  - Superconducting microwave Fabry Perot cavity
    - A nearly ideal photon box
      - T_c=0.13 s at 0.8 K: a macroscopic time interval
      - Q=4.2 10^{10}, F=4.6 10^9



Mirror technology

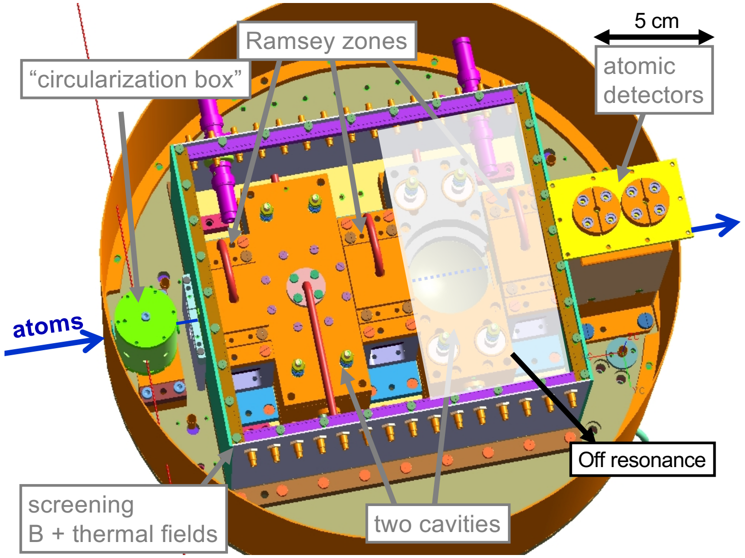
- Copper substrates
  diamond machining
    ~shape accuracy 300 nm ptv
    ~rugosity 10 nm
  Toroidal surface ➔ single mode
- 12 µm Niobium layer
  Cathode plasma sputtering
  CEA, Saclay
  [E. Jacques, B. Visentin, P. Bosland]

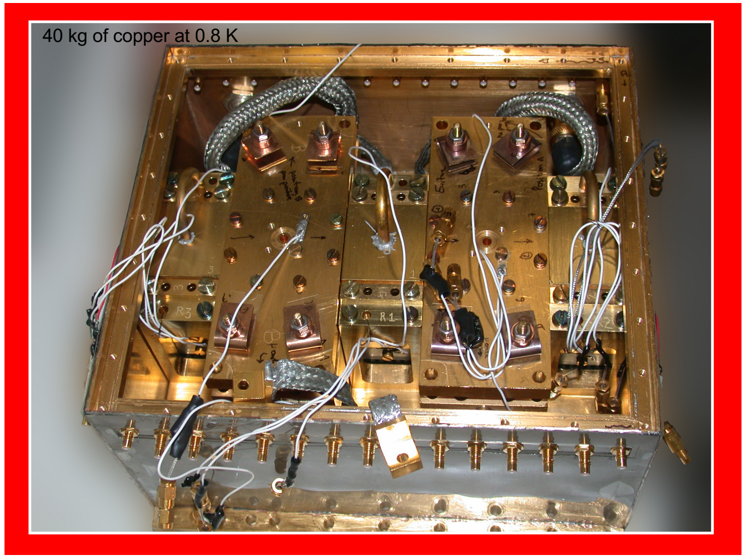
S. Kuhr et al, APL, **90**, 164101



Experimental set-up

RMP **73**, 565



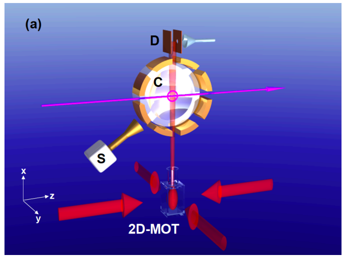


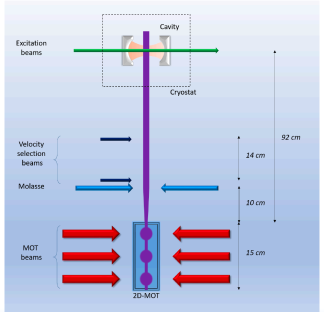
40 kg of copper at 0.8 K

A CQED experiment with slow atoms

F. Assemat et al PRL in print arXiv:1905.05247

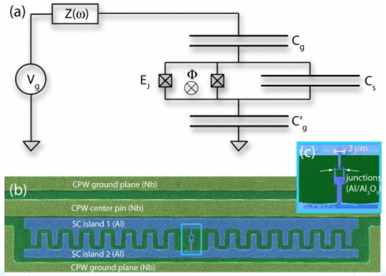
- Thermal beam
  - Atomic velocity 200-500 m/s, interaction time limited to 80 µs
  - A strong limitation for some experiments
- A cavity QED experiment in an atomic fountain configuration
  - 10 m/s velocity, very long interaction times





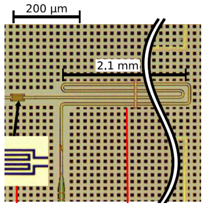
The tools of circuit QED

- Atoms
  - Superconducting circuits based on Josephson junctions
  - Most popular example: the transmon
    - A weakly non linear harmonic oscillator
    - Long relaxation times up to tens of µs

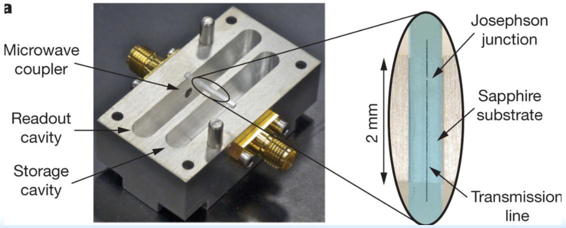


The tools of circuit QED

- Cavities
  - Superconducting stripline resonators on a chip



  - 3D Pill-box superconducting cavities

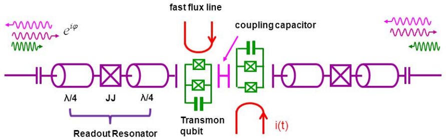


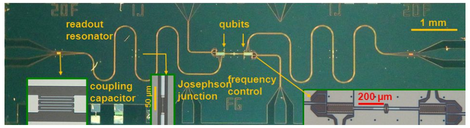
The tools of circuit QED

- A complete set-up

Example : capacitively coupled transmons with individual readout

(Saclay, 2011)





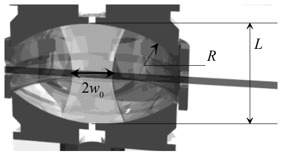
- A thriving field worldwide
  - Schoelkopf, Devoret (Yale); Martinis (Santa Barbara); Wallraff (ETH) Esteve (CEA); Nakamura (NTT)…

Circuit and Cavity QED

- Two totally different experimental approaches
  - Tools of AMO physics or solid-state
  - Both in the microwave (5-50GHz domain)
- Very different orders of magnitude
  - Much higher atom-field coupling in circuit QED but much faster dissipative process
  - Much longer atom-cavity interaction time in circuit QED
- But, *mutatis mutandis,* similar capabilities and achievements.
  - All the discussions in these lecture apply to the cavity QED context but can be immediately transposed to circuit QED
    - And, in a large part to trapped ions

Cavity mode volume

- A Fabry perot resonator with a Gaussian standing-wave mode
  - Directly described by the formalism of the previous lectures
    - Normalization: *f* is 1 at the field maximum



- Defines the cavity mode volume as

$$\mathcal{V} = \int |\mathbf{f}(\mathbf{r})|^2 \, d^3\mathbf{r} = \frac{\pi w_0^2 L}{4}$$

$$\mathcal{V} = 0.7 \text{ cm}^3, \, \mathcal{E}_0 = 1.5 \text{ V/m}$$

Taking into account atomic motion: effective interaction time

- Real atoms cross gaussian mode: vacuum Rabi frequency is a function of time

$$\Omega_0 f(vt) \qquad f(vt) = e^{-v^2t^2/w_0^2}$$

- Simple expressions only in resonant and dispersive cases
  - Resonant case $\widetilde{H}(t) = f(vt)\widetilde{H}(0)$
    - Interaction from t_i to t_f

$$U_r = \exp\left[-(i/\hbar)\int_{t_i}^{t_f} \widetilde{H}(t)\,dt\right] = \exp\left[-(i/\hbar)\widetilde{H}(0)t_i^r\right]$$

- For a full cavity transity

$$t_i^r = \int f(vt)\,dt = \sqrt{\pi}\frac{w_0}{v} \; .$$

- Replace everywhere $\Omega_n(\mathbf{r})$ by $\Omega_0\sqrt{n+1}$ and use effective interaction time

Taking into account atomic motion

- Dispersive case
  - Define effective Hamiltonian, proportional to |f|^2

$$H_{eff}(\mathbf{r}) = \hbar s_0 |f(\mathbf{r})|^2 \left[\sigma_z \left(a^\dagger a + \tfrac{1}{2}\right) + \tfrac{1}{2}\mathbb{1}\right] \qquad s_0 = \frac{\Omega_0^2}{4/\Delta_c}$$

$$U^d = \exp\left[-(i/\hbar)H_{eff}(0)t_i^d\right]$$

- Full cavity transit

$$t_i^d = \int f^2(vt)\,dt = \sqrt{\frac{\pi}{2}}\frac{w_0}{v}$$

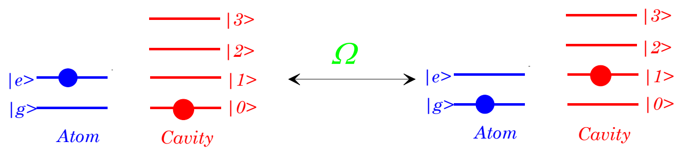
- Use the effective interaction time and the shifts at cavity centre
- Note that resonant and non-resonant effective interaction times are not equal

Outline of this lecture

- Introduction
- Tools of cavity QED
- Resonant interaction
  - Vacuum Rabi oscillation
  - Rabi in a mesoscopic field
- Dispersive interaction
- Perspectives

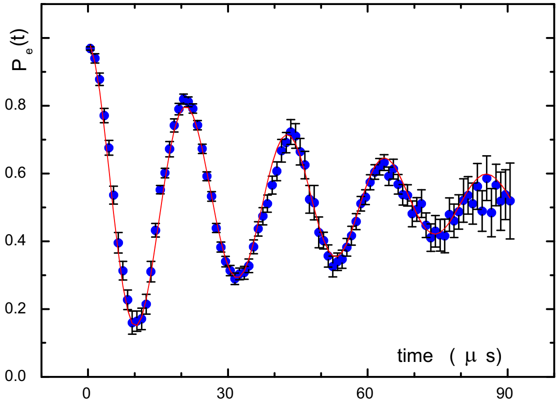
Vacuum Rabi oscillations

- The simplest situation: initial state $|e,0\rangle$



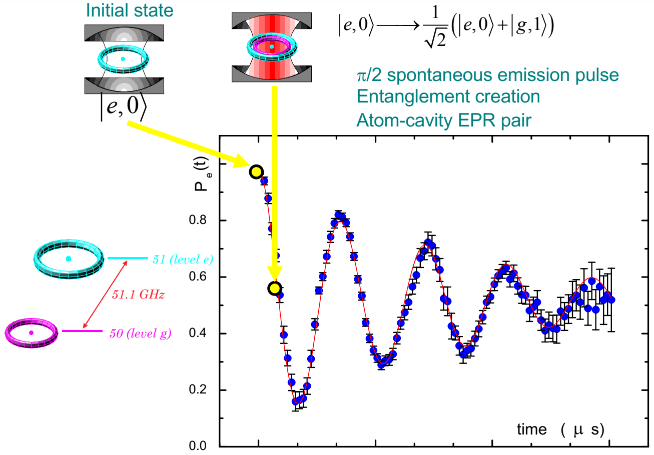
- Periodic exchange of one oscillation quantum between the atom an the cavity at the `vacuum Rabi frequency’ Ω_0.
- Time counterpart of the `vacuum Rabi splitting’
- Observable only in the strong coupling regime, when oscillation frequency is much higher than the dissipation rates

An early experiment



Brune et al, PRL **76,** 1800 (96)

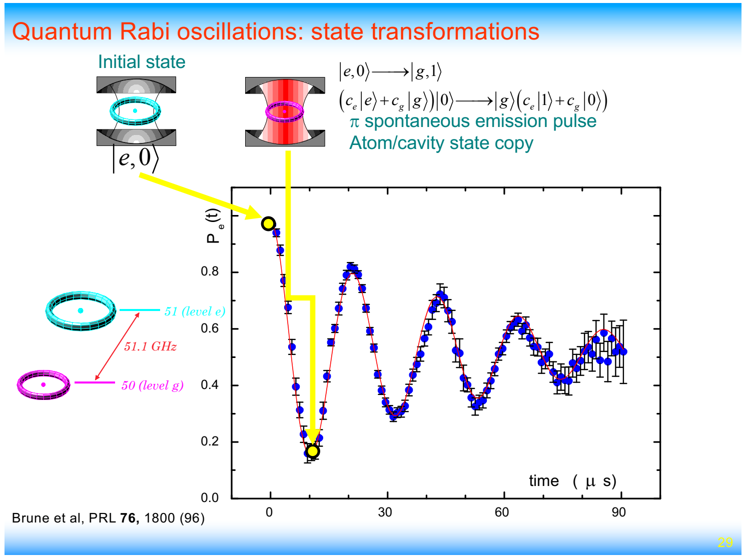
Quantum Rabi oscillations: state transformations



$$|e,0\rangle \longrightarrow \frac{1}{\sqrt{2}}\left(|e,0\rangle + |g,1\rangle\right)$$

π/2 spontaneous emission pulse
Entanglement creation
Atom-cavity EPR pair

Brune et al, PRL **76,** 1800 (96)



Quantum Rabi oscillations: state transformations

Initial state

$|e,0\rangle$

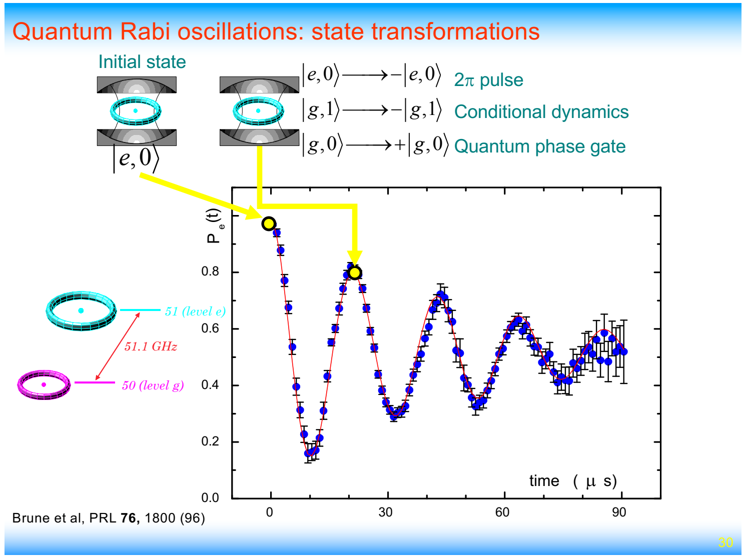
$$|e,0\rangle \longrightarrow |g,1\rangle$$

$$(c_e|e\rangle + c_g|g\rangle)|0\rangle \longrightarrow |g\rangle(c_e|1\rangle + c_g|0\rangle)$$

π spontaneous emission pulse

Atom/cavity state copy

Brune et al, PRL **76,** 1800 (96)



Quantum Rabi oscillations: state transformations

Initial state

$|e,0\rangle$

$$|e,0\rangle \longrightarrow -|e,0\rangle$$ 2π pulse

$$|g,1\rangle \longrightarrow -|g,1\rangle$$ Conditional dynamics

$$|g,0\rangle \longrightarrow +|g,0\rangle$$ Quantum phase gate

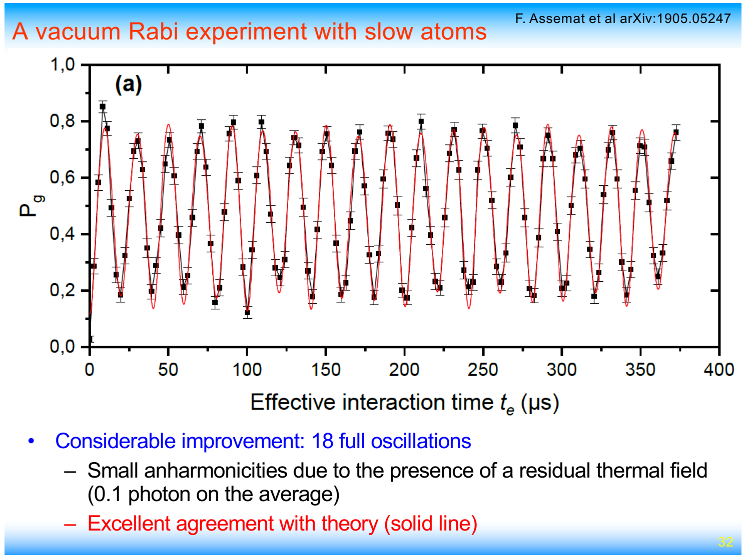
Brune et al, PRL **76,** 1800 (96)

Three "stitches" to "knit" quantum entanglement

Combine elementary transformations to create entangled states

an early exploration of quantum logics

- State copy with a π pulse
  - Quantum memory : PRL **79**, 769 (97)
- Creation of entanglement with a π/2 pulse
  - EPR atomic pairs : PRL **79**, 1 (97)
- Quantum phase gate based on a 2π pulse
  - Quantum gate : PRL **83**, 5166 (99)
  - Absorption-free detection of a single photon: Nature **400**, 239 (99)
- Entanglement of three systems (six operations on four qubits)
  - GHZ Triplets : Science **288**, 2024 (00)
- Entanglement of two radiation field modes
  - Phys. Rev. A **64**, 050301 (2001)
- Direct entanglement of two atoms in a cavity-assisted collision
  - Phys. Rev. Lett. **87**, 037902 (2001)



A vacuum Rabi experiment with slow atoms

F. Assemat et al arXiv:1905.05247

- Considerable improvement: 18 full oscillations
  - Small anharmonicities due to the presence of a residual thermal field (0.1 photon on the average)
  - Excellent agreement with theory (solid line)

Outline of this lecture

- Introduction
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  - Vacuum Rabi oscillation
  - Rabi in a mesoscopic field
- Dispersive interaction
- Perspectives

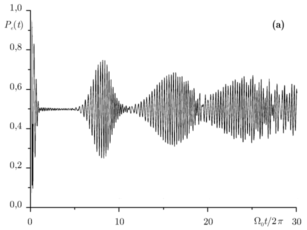
Rabi oscillation in a mesoscopic coherent field

- Intermediate regime of a few tens of photons.
  - A simple theoretical problem $|\alpha\rangle = \sum_n c_n|n\rangle$;

$$|\Psi(t)\rangle = \sum_n c_n \cos\frac{\Omega_0\sqrt{n+1}t}{2}|e,n\rangle + c_n \sin\frac{\Omega_0\sqrt{n+1}t}{2}|g,n+1\rangle \ ;$$

$$P_e(t) = \sum_n p_c(n)\frac{1+\cos\Omega_0\sqrt{n+1}t}{2} \ .$$

  - A surprisingly complex behavior
    - Computed for15 photons



Collapse and revival

- Collapse:
  - dispersion of field amplitudes due to dispersion of photon number

$$t_c \approx \pi/\Omega_0 \ .$$

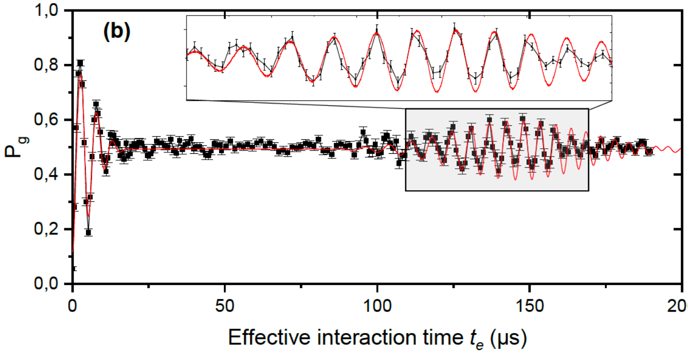
- Revival:
  - rephasing of amplitudes at a finite time such that oscillations corresponding to n and n+1 come back in phase

$$t_r \approx \frac{4\pi}{\Omega_0}\sqrt{n} \ .$$

  - Revival is a genuinely quantum effect
  - Get a better understanding

Observation with slow atoms

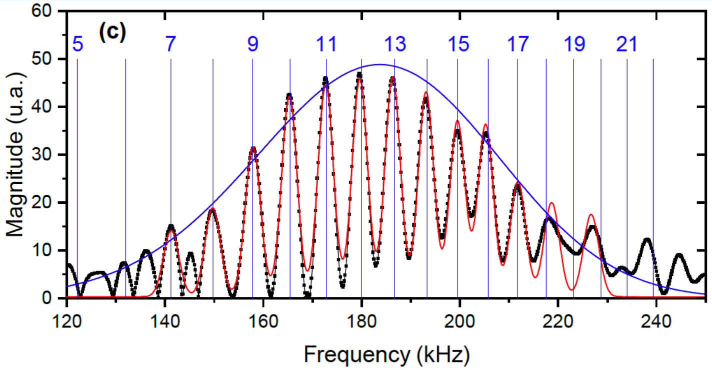
F. Assemat et al PRL in print arXiv:1905.05247



- Clear revivals in a 13 photon field !
  - Excellent agreement with a numerical simulation taking into account a few experimental imperfections (detection errors, residual thermal field…)

Field amplitude quantization

F. Assemat et al PRL in print arXiv:1905.05247



- Revival signal Fourier Transform reveals discrete frequencies
  - Direct evidence of field quantization
  - Direct measurement of photon number distribution

An insightful quasi-exact solution

J. Gea-banacloche PRL 65, 3385 (1990)
V. Buzek et al. PRA 45, 8190 (1992)

$$\left|\Psi(t)\right\rangle = \frac{1}{\sqrt{2}}\left[\left|\Psi_a^+(t)\right\rangle\left|\Psi_c^+(t)\right\rangle + \left|\Psi_a^-(t)\right\rangle\left|\Psi_c^-(t)\right\rangle\right]$$

$$\left|\Psi_a^\pm\right\rangle = \frac{1}{\sqrt{2}} e^{\pm i\Omega_0\sqrt{n}t/2}\left[e^{\pm i\Phi}\left|e\right\rangle \mp i\left|g\right\rangle\right] \qquad \Phi = \frac{\Omega_0 t}{4\sqrt{n}}$$

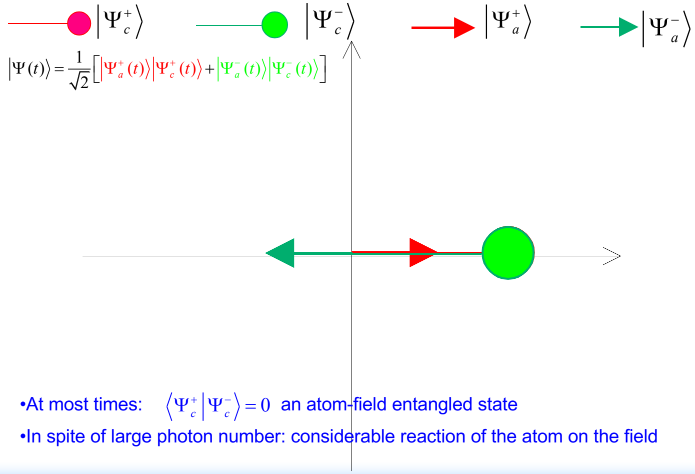
- Atomic states slowly ($\bar{n}$ times slower than Rabi oscillation) rotating in the equatorial plane of the Bloch sphere

$$\left|\Psi_c^\pm\right\rangle = e^{\mp i\Omega_0\sqrt{n}t/4}\left|\alpha e^{\pm i\Phi}\right\rangle$$

- A slowly rotating field state in the Fresnel plane

- Graphical representation of the joint atom-field evolution in a plane
- t=0:
  - both field states coincide with original coherent state
  - Atomic states are the classical eigenstates

Atom-field states evolution



$$\left|\Psi(t)\right\rangle = \frac{1}{\sqrt{2}}\left[\left|\Psi_a^+(t)\right\rangle\left|\Psi_c^+(t)\right\rangle + \left|\Psi_a^-(t)\right\rangle\left|\Psi_c^-(t)\right\rangle\right]$$

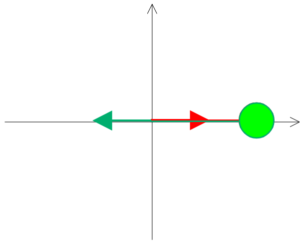
- At most times: $\left\langle\Psi_c^+\middle|\Psi_c^-\right\rangle = 0$ an atom-field entangled state
- In spite of large photon number: considerable reaction of the atom on the field

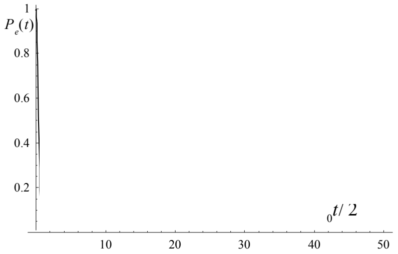
Link with Rabi oscillation

$$\left|\Psi(t)\right\rangle = \frac{1}{\sqrt{2}}\left[\left|\Psi_a^+(t)\right\rangle\left|\Psi_c^+(t)\right\rangle + \left|\Psi_a^-(t)\right\rangle\left|\Psi_c^-(t)\right\rangle\right]$$

Rabi oscillation: quantum interference between $\left|\Psi_a^+\right\rangle$ and $\left|\Psi_a^-\right\rangle$

- Contrast vanishes when $\left\langle\Psi_c^+\middle|\Psi_c^-\right\rangle = 0$:
  - A direct link between Rabi collapse and complementarity





Generation of Schrödinger cat states

- At half revival, the atom is disentangled
  - Field left in a superposition of two coherent amplitudes with opposite phases

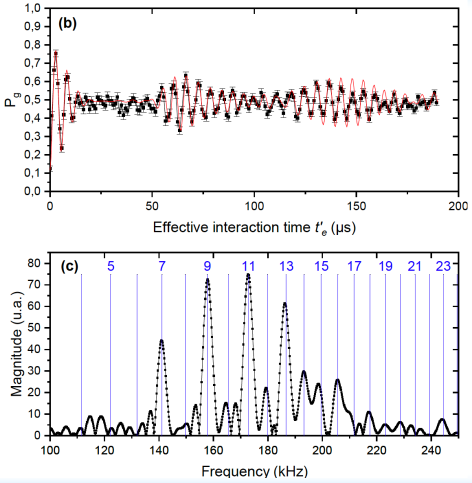
$$|\psi_{cat}\rangle = e^{i\pi\overline{n}} |i\beta\rangle - |-i\beta\rangle$$

  - A Schrödinger cat states of the field (and a large one)
  - Resonant interaction leads to the fastest preparation of such cat states
- Check the cat:
  - At half revival, reset the atom in state e and record its Rabi oscillation in the cat (with a small deterministic translation in phase space)

The cat's fingerprint

F. Assemat et al PRL in print arXiv:1905.05247

- Rabi oscillations in the cat generated at half revival
  - An extra early revival
  - Photon number distribution contains only odd photon numbers
    - A direct evidence of a cat state

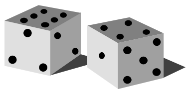


Outline of this lecture

- Introduction
- Tools of cavity QED
- Resonant interaction
- Dispersive interaction
  - Ideal QND measurement of photon number
  - Quantum Zeno effect
  - Fock states
  - Complementarity and Schrödinger cats
  - Quantum feedback
- Perspectives

Ideal quantum measurement of the photon number

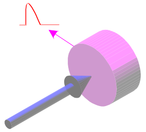
- Quantum discontinuity
  - Not all measurement results allowed
    - eg: number of photons is an integer
- Quantum indeterminacy
  - Predict only the probabilities of possible outcomes
  - Result of a measurement intrinsically random
    - God is playing dice
    - e.g. Poisson photon number distribution in a classical laser pulse



- Repeatability
  - Repetition of ideal measurements of a constant of motion always give the same result
    - Projection postulate.
    - Repeated measurements of the photon number always give the same result

Ideal and real quantum measurements

- Most quantum measurements are far from ideal
  - e.g. Photodetection (counting photons)
    - measurement of light field energy
      - quantized result: number of photons
      - statistical: photon number statistics
      - repeatable?



    - Photodetectors absorb incoming photons
      - A second detection always gives zero: impossible to 'see' the same photon twice
      - The field state is demolished by the detection
    - This demolition is not a requirement of quantum mechanics

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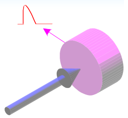
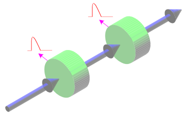
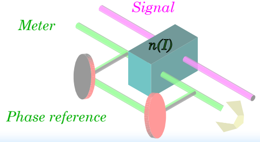
Ideal photon number counting



- Most quantum measurements are far from being projective
  - Light detection:
    - Photons are destroyed when detected

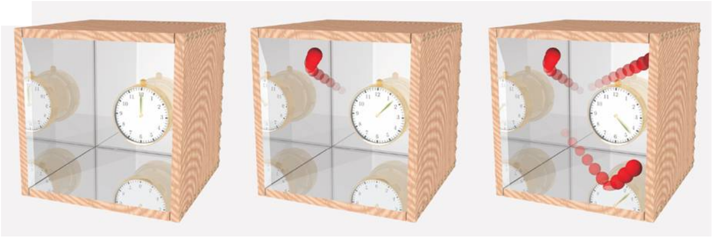


- Quantum non-demolition measurements (Braginsky, 70s)
  - A transparent photocounter
    - 'see' the same photon twice
    - Should allow observation of the quantum jumps of light



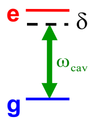
- Realized in the optical domain (Grangier et al, Nature, 396, 537)
  - no single photon resolution
    - weak non-linearity
  - propagating fields:
    - repetition difficult

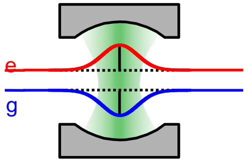
Yet another gedankenexperiment



- A clock whose ticking rate is determined by the number of photons in a box
- The final clock hand's position directly measures the photon number
  - Box: a superconducting millimetre wave cavity
  - Clock: a single circular Rydberg atom

Dispersive atom-field interaction

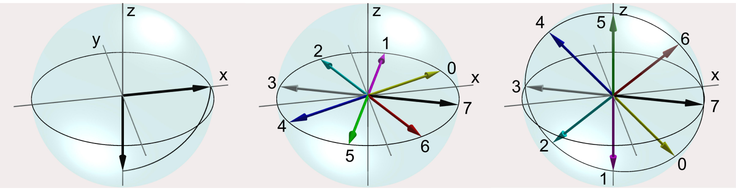




- Atomic frequency shift inside the cavity
  - Light and Lamb shifts:
    - An atomic clock ticking rate modification $\delta\omega_{eg} = s_0(2n+1)$
    - A phase shift of the atomic coherence $\phi_0(n+1/2)$
- Adiabatic coupling in and out of the atom-cavity interaction
  - negligible spurious absorption rate ($<10^{-4}$ for $\delta \sim \Omega$)

A pictorial representation of the interaction

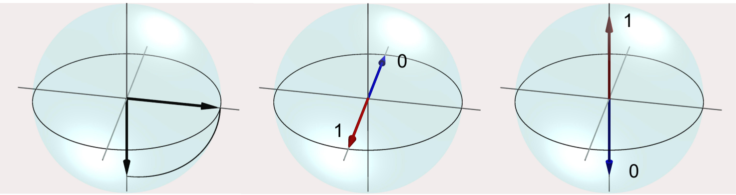
- Evolution of the atomic state on the Bloch sphere.
  - $\pi/4$ phase shift per photon



- In general non-orthogonal final atomic states correspond to different photon numbers: A single atom does not tell all the story
- A simple case: π phase shift per photon and 0/1 photon (a 'qubit' situation)

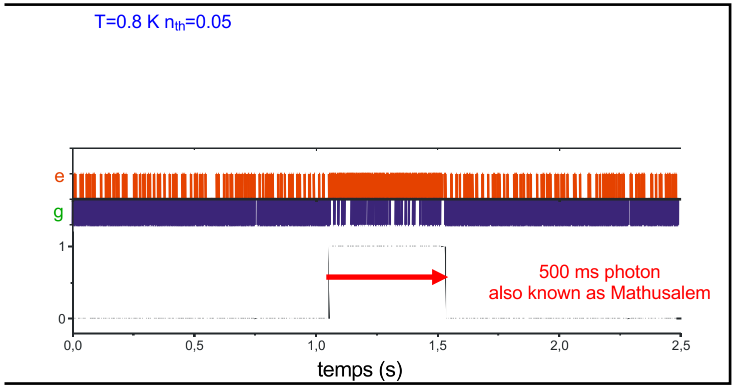
The single photon case

- A zero or one-photon field
  - π phase shift per photon



- Two orthogonal final atomic states
  - in principle, a single atomic detection unambiguously tells the photon number.

Birth, life and death of a photon

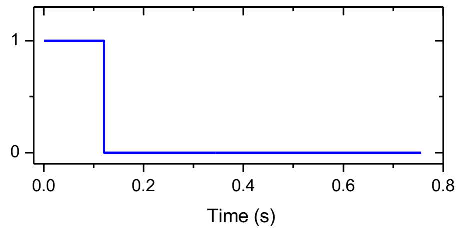


500 ms photon
also known as Mathusalem

Gleyzes et al, Nature, **446**, 297 (2007)

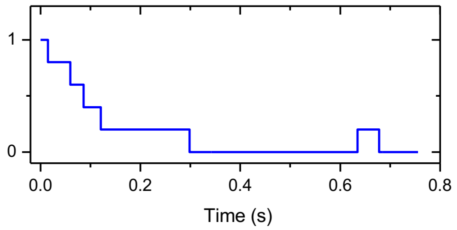
|n=1> Lifetime

1 sequence :



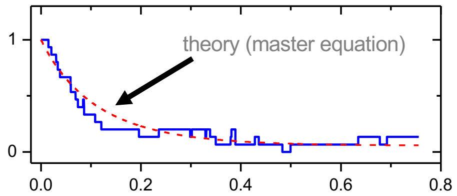
|n=1> Lifetime

5 sequences :



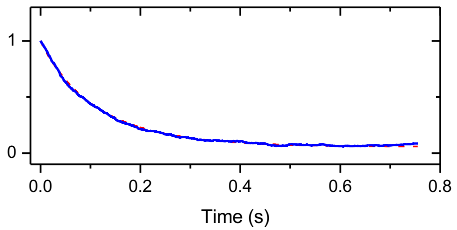
|n=1> Lifetime

15 sequences :



|n=1> Lifetime

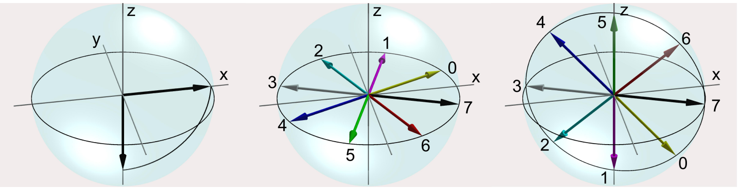
904 sequences :



Excellent agreement with the quantum predictions (no adjustable parameter)

Counting from 0 to 7

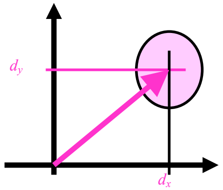
- $\pi/4$ phase shift per photon
  - Evolution of the atomic state on the Bloch sphere.



  - In general non-orthogonal final atomic states correspond to different photon numbers: A single atom does not tell all the story

Photon counting by information accumulation

- One atom exits cavity with a spin direction correlated to *n*
- QND interaction: N atoms exit cavity with the same spin direction correlated to *n*
  - Entanglement of the photon number with a mesoscopic atomic sample
- Split atomic sample in two parts
  - On *N*/2 atoms, measure S_x
  - On *N/2* atoms, measure S_y
  - Estimate spin direction with $1/\sqrt{N}$ uncertainty



S. Gleyzes et al. Nature **446**, 297, C. Guerlin et al. Nature **448**, 889

"Forward" estimation of the photon number at time *t*

- Density operator ρ including all available information from 0 to t
  - Updated according to each atomic detection result

$$\rho_{p-1}^f \longrightarrow \rho_p^f = \frac{M_j \rho_{p-1}^f M_j^\dagger}{\pi_j(\phi_r|\rho_{p-1}^f)}$$

    - Measurement operators

$$M_g = \sin\left[\frac{\phi_r + \phi_0(N+1/2)}{2}\right]$$

$$M_e = \cos\left[\frac{\phi_r + \phi_0(N+1/2)}{2}\right]$$

$$\pi_j(\phi_r|\rho) = \mathrm{Tr}\left(M_j \rho M_j^\dagger\right)$$

  - Updated according to cavity relaxation between detections
    - Liouvillian evolution

A Bayesian inference process

- Photon number distribution
  - Relaxation and measurement operators diagonal in the Fock states basis
    - Updated according to atomic detection results

$$P_p^f(n) = \frac{\pi_j(\phi_r|n)}{\pi_j(\phi_r|\rho)} P_{p-1}^f(n)$$

    – Detection probabilities:

$$\pi_j(\phi_r|\rho) = \sum P(n)\pi_j(\phi_r|n)$$

$$\pi_e(\phi_r|n) = 1 - \pi_g(\phi_r|n) = \frac{1}{2}\left(1 + \cos\left[\phi_r + \phi_0(n+1/2)\right]\right)$$

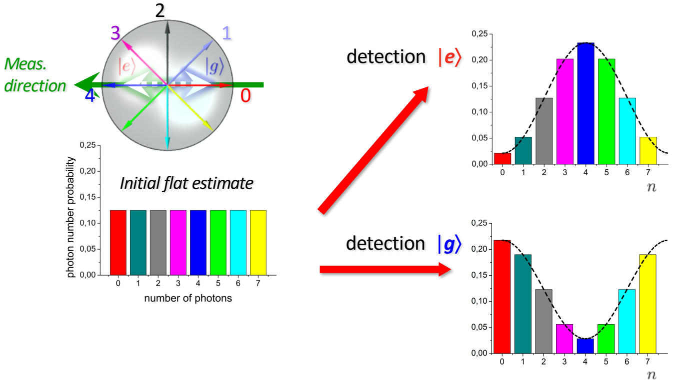
    - Updated according to cavity relaxation

$$\frac{dP^f(n,t)}{dt} = \sum_m K_{n,m} P^f(m,t)$$

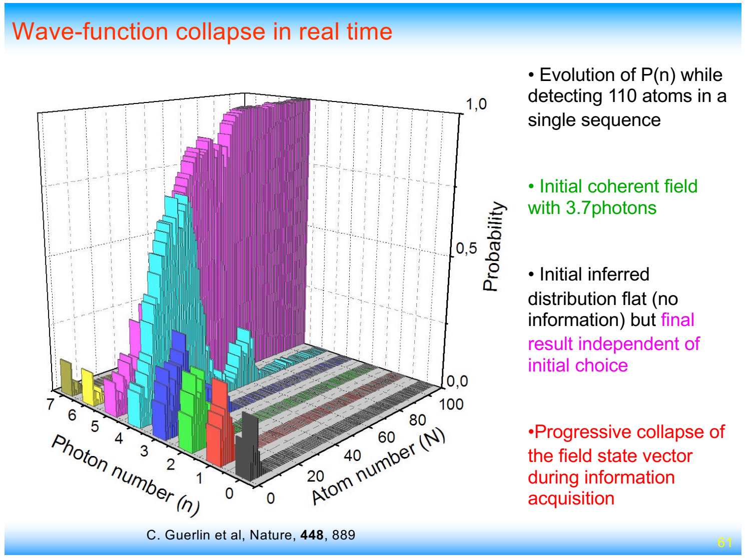
- A Bayesian inference of P(n) by photon decimation, proceeding forward in time
  - About 8² atoms required to count from 0 to 7

Single atom detection

Each detection brings partial information on the photon number



To speed-up convergence, the measurement phase ϕ_r is randomly chosen for each atom among the four values corresponding to atomic states



Wave-function collapse in real time

- Evolution of P(n) while detecting 110 atoms in a single sequence
- Initial coherent field with 3.7photons
- Initial inferred distribution flat (no information) but final result independent of initial choice
- Progressive collapse of the field state vector during information acquisition

C. Guerlin et al, Nature, **448**, 889

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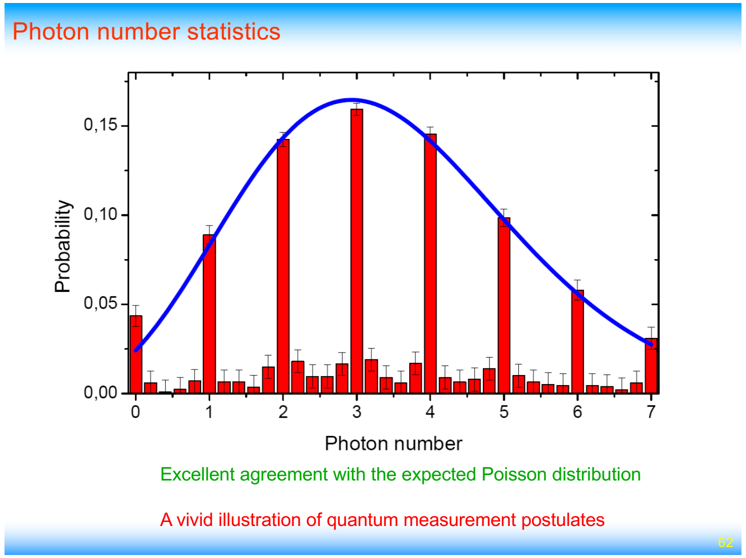

Photon number statistics

Excellent agreement with the expected Poisson distribution

A vivid illustration of quantum measurement postulates

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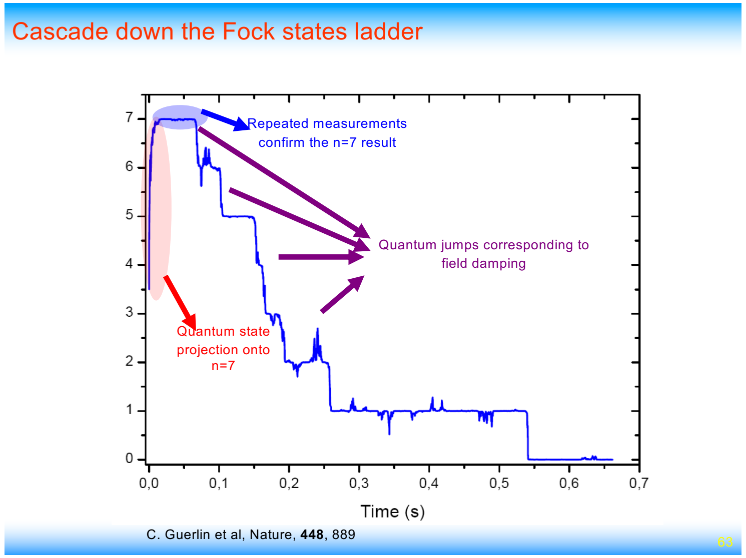

Cascade down the Fock states ladder

Repeated measurements confirm the n=7 result

Quantum jumps corresponding to field damping

Quantum state projection onto n=7

C. Guerlin et al, Nature, **448**, 889

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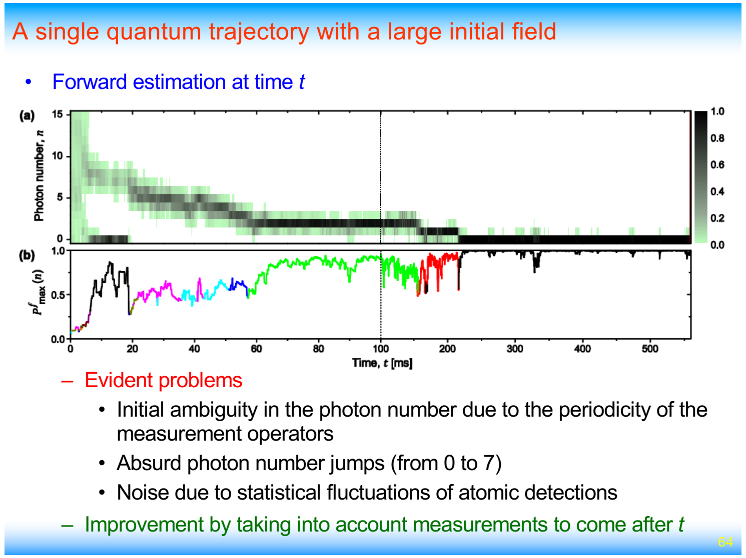

A single quantum trajectory with a large initial field

- Forward estimation at time *t*

- Evident problems
  - Initial ambiguity in the photon number due to the periodicity of the measurement operators
  - Absurd photon number jumps (from 0 to 7)
  - Noise due to statistical fluctuations of atomic detections
- Improvement by taking into account measurements to come after *t*

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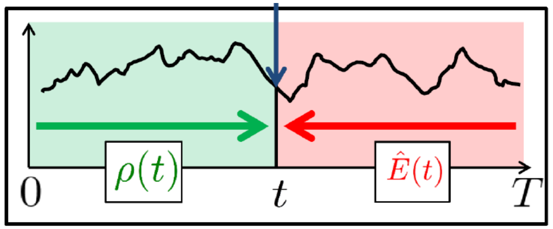
The Past Quantum State approach

- A posteriori estimation of the photon number at *t* based on all available information, gathered from 0 to *t* AND from *t* to *T*
  - From the journalist's to the historian's perspective
- A quantum formalism (S. Gammelmak et al. PRL 111, 160401)
  - The Past quantum state



  - Best estimate for the results of a quantum measurement at *t* based on the density matrix ρ computed forward in time AND on an effect matrix *E* computed backwards in time

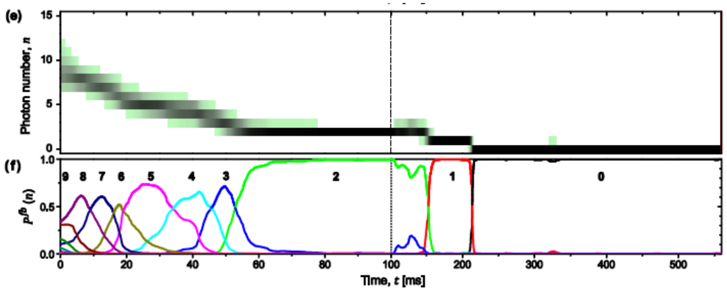
Forward-backward estimation

- For diagonal measurement/relaxation operators

$$P^{fb}(n,t) = \frac{P^f(n,t)P^b(n,t)}{\sum_m P^f(m,t)P^b(m,t)}$$

  - PQS reduces to the forward/backward smoothing algorithm, which can be safely used in this quantum context
  - P(n) is the product of two photon number distributions computed forward and backward in time.
- Backwards estimation
  - Flat distribution at *T*
  - Same measurement operators
  - 'inverse' relaxation (annihilation and creation operators exchanged)
    - Exponential growth of the photon number

PQS estimation



- Ambiguities lifted
  - Measurement of photon number beyond the intrisic periodicity of atomic signal
- Considerable noise reduction
  - All estimations take into account ALL available information

T. Rybarczyk et al., PRA 91 062116

Outline of this lecture

- Introduction
- Tools of cavity QED
- Resonant interaction
- Dispersive interaction
  - Ideal QND measurement of photon number
  - Quantum Zeno effect
  - Fock states
  - Complementarity and Schrödinger cats
  - Quantum feedback
- Perspectives

Quantum Zeno effect

- A watched kettle never boils
  - coherent evolution of a system and frequently repeated quantum measurements
    - a quantum jumps evolution between eigenstates of the measured quantity
    - an evolution much slower than without measurements
    - no evolution at all in the limit of zero delay between measurements
  - No Zeno effect for incoherent relaxation processes

Quantum Zeno effect

- A simple description of the Zeno effect
  - A quantum system initially in |0> evolves under the action of the hamiltonian *V* during time *t*.
  - During this time, *n* measurements of an observable *O* with the non-degenerate eigenstate |0> are performed, at times *t/n*, *2t/n*…
  - At *t/n* probability for finding |0> is

$$\Pi_0\left(\frac{t}{n}\right) = 1 - \frac{\Delta^2 V}{\hbar^2}\frac{t^2}{n^2} + \cdots \quad ; \quad \Delta^2 V = \sum_{i\neq 0}\left|\langle 0|V|i\rangle\right|^2$$

    - A quadratic function of the time interval *t/n*
  - Final probability for finding |0>:

$$\Pi_0^{(n)}(t) = \left[1 - \frac{\Delta^2 V}{\hbar^2}\frac{t^2}{n^2} + \cdots\right]^n = 1 - \frac{\Delta^2 V}{\hbar^2}\frac{t^2}{n} + \cdots \xrightarrow[n\to\infty]{} 1$$

    - 1 if the time interval between measurements is close to zero.
    - Efficient inhibition of coherent evolution

Quantum Zeno effect

- Coherent evolution
  - Probability for leaving |0> quadratic in *t/n*
    - Efficient inhibition of coherent evolution
- Incoherent evolution (relaxation)
  - Probability for leaving |0> in the first step $\Gamma t/n$ (exponential decay)
  - Final probability for staying in |0> (assume Γt<<1):

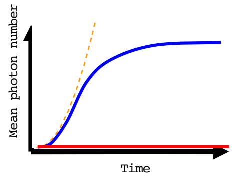
$$\left(1 - \Gamma\frac{t}{n}\right)^n \approx 1 - \Gamma t$$

    - Same decay without measurements
  - Zeno effect does not affect relaxation processes
    - Unless measurements frequently repeated on the scale of the environment's correlation time

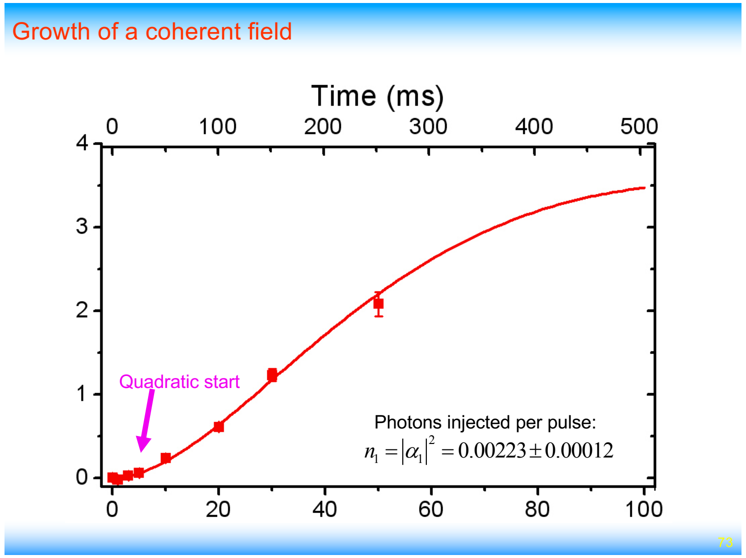
Quantum Zeno effect

- Coherent evolution: injection of a coherent field by a classical source

–Repeated injection of phase coherent pulses: an amplitude varying linearly with the number of injections (photon number varies quadratically).



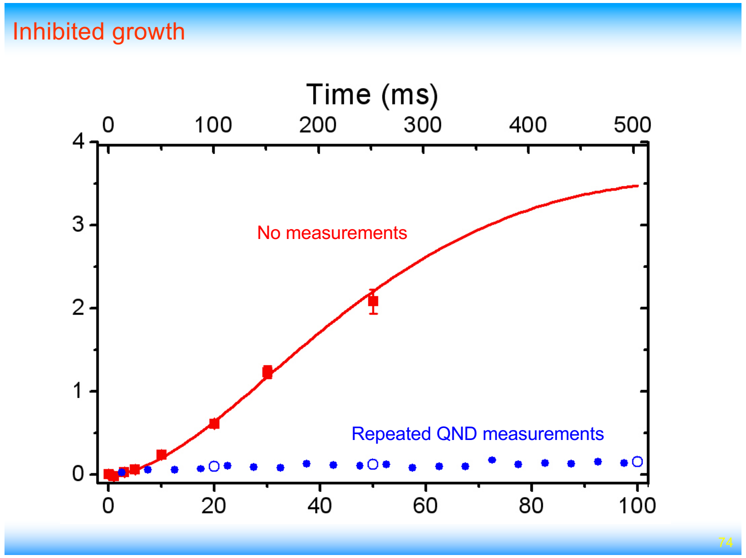
Principle of the experiment: perform QND measurements of photon number between two pulses



Growth of a coherent field

Quadratic start

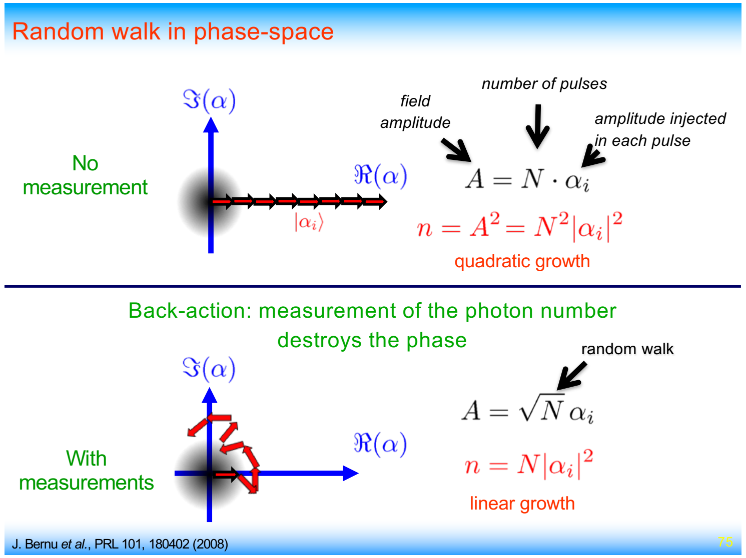
Photons injected per pulse:
$n_1 = |\alpha_1|^2 = 0.00223 \pm 0.00012$



Inhibited growth

No measurements

Repeated QND measurements



Random walk in phase-space

No measurement

field amplitude

number of pulses

amplitude injected in each pulse

$$A = N \cdot \alpha_i$$

$$n = A^2 = N^2|\alpha_i|^2$$

quadratic growth

Back-action: measurement of the photon number destroys the phase

With measurements

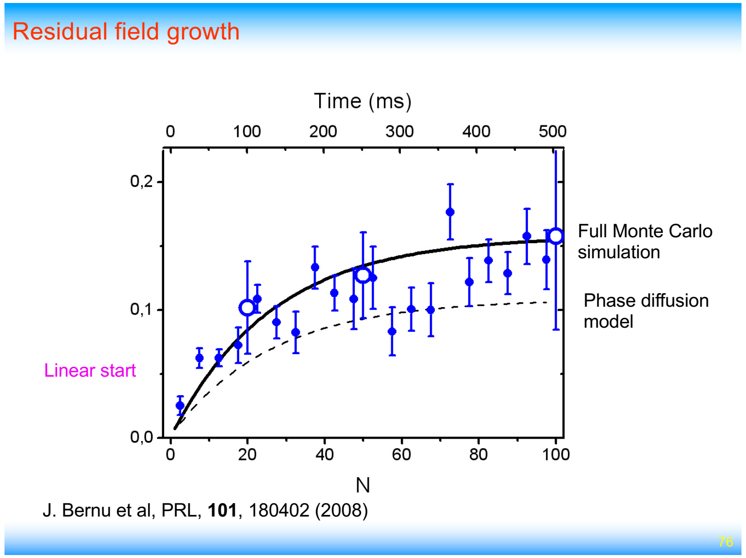
random walk

$$A = \sqrt{N}\,\alpha_i$$

$$n = N|\alpha_i|^2$$

linear growth

J. Bernu *et al.*, PRL 101, 180402 (2008)



Residual field growth

Full Monte Carlo simulation

Phase diffusion model

Linear start

J. Bernu et al, PRL, **101**, 180402 (2008)

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Fock state preparation

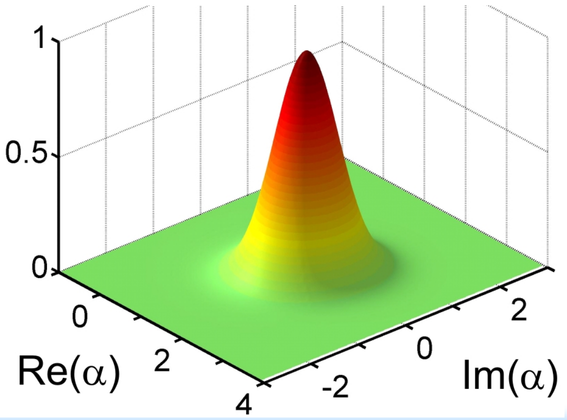
- Ideal projective measurement
  - After measurement the field is in a photon number state
  - Prepare all Fock states from 0 to 7
    - Not an easy task in quantum optics
- Check the produced state ?
  - Full measurement of the cavity quantum state
    - Also based on the QND interaction
    - Get all the density operator describing the field state
  - Present it in terms of the field's Wigner function:
    - A 'wavefunction' in the phase plane (Fresnel plane)
    - A quasiprobability distribution for the complex field amplitude.

S. Deléglise et al, Nature, **455**, 510 (2008)

Coherent states

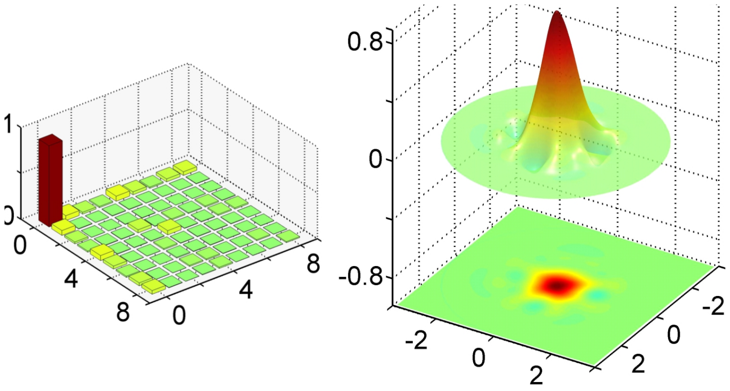
- A coherent state with 2.5 photons

F=0.98



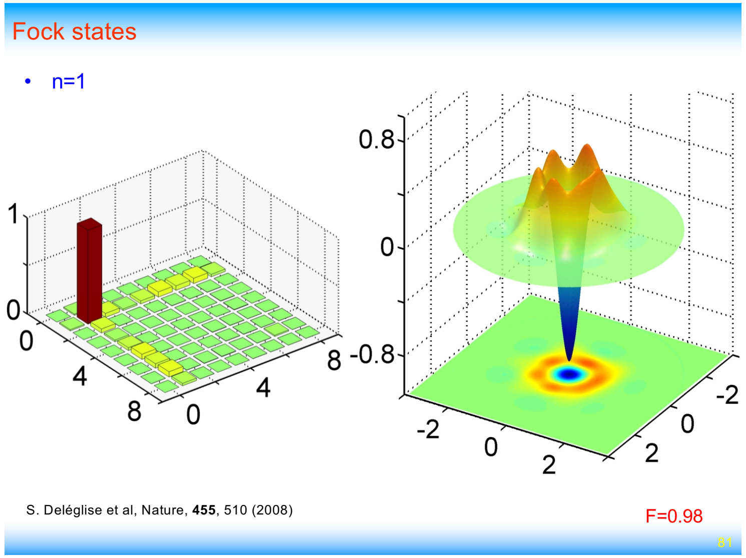
Fock states

- n=0



S. Deléglise et al, Nature, **455**, 510 (2008)

F=0.89

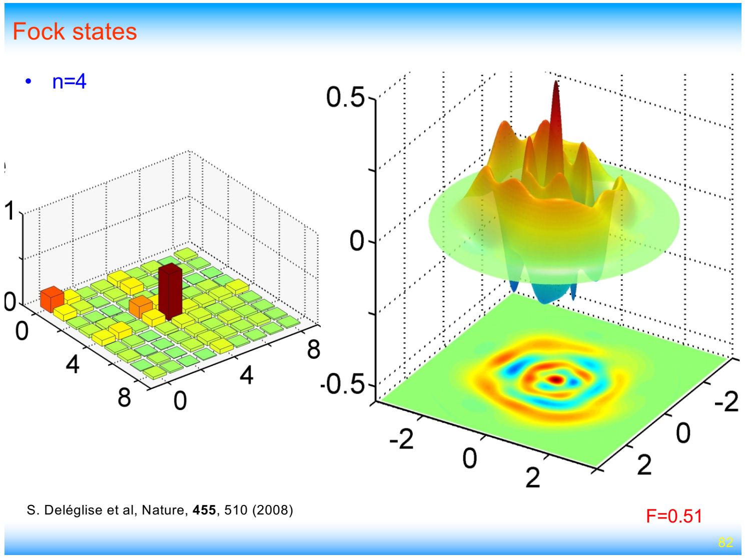


Fock states

- n=1

S. Deléglise et al, Nature, **455**, 510 (2008)

F=0.98

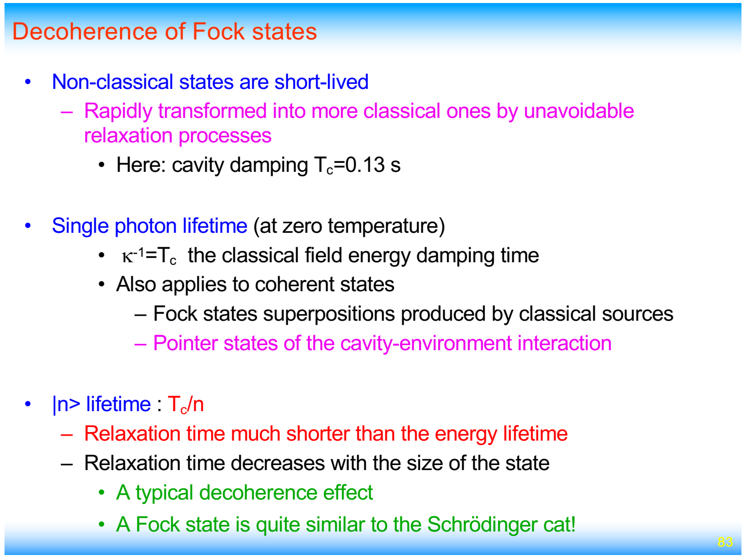


Fock states

- n=4

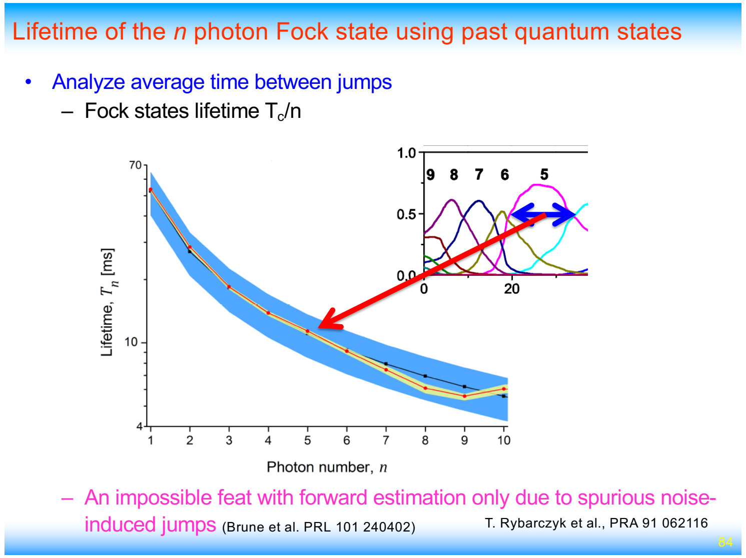
S. Deléglise et al, Nature, **455**, 510 (2008)

F=0.51



Decoherence of Fock states

- Non-classical states are short-lived
  - Rapidly transformed into more classical ones by unavoidable relaxation processes
    - Here: cavity damping T_c=0.13 s
- Single photon lifetime (at zero temperature)
  - $\kappa^{-1}=T_c$ the classical field energy damping time
  - Also applies to coherent states
    - Fock states superpositions produced by classical sources
    - Pointer states of the cavity-environment interaction
- |n> lifetime : T_c/n
  - Relaxation time much shorter than the energy lifetime
  - Relaxation time decreases with the size of the state
    - A typical decoherence effect
    - A Fock state is quite similar to the Schrödinger cat!



Lifetime of the *n* photon Fock state using past quantum states

- Analyze average time between jumps
  - Fock states lifetime T_c/n

  - An impossible feat with forward estimation only due to spurious noise-induced jumps (Brune et al. PRL 101 240402)

T. Rybarczyk et al., PRA 91 062116

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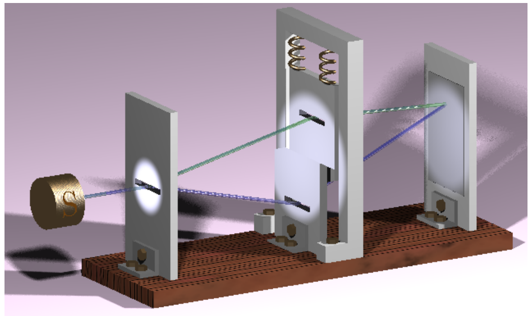
Bohr's thought experiment on complemetarity

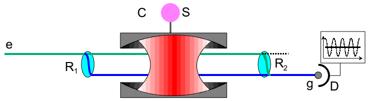
- Complementarity (From Einstein-Bohr at the 1927 Solvay congress)



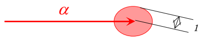
  - Moving slit records the trajectory of the particle in the interferometer
    - Which path information but no fringes
    - Or no which path but fringes
  - Wave and particle are complementary aspects of the quantum object.

Cavity field as a which path detector

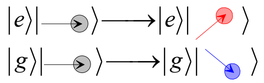
- Insert non-resonant cavity inside the interferometer



  - Cavity contains initially a mesoscopic coherent field



  - The two atomic levels produce opposite phase shifts of the cavity field



- Field amplitude is the 'needle' of a 'meter' pointing towards atomic state
  - Prototype of a quantum measurement
  - Provides a which-path information and should erase the fringes

Action of an atom on a coherent field in the dispersive regime

- Effective Hamiltonian

$$H_{eff} = \hbar s_0 \left[\sigma_Z (a^\dagger a + \frac{1}{2}) + \frac{1}{2}\mathbb{1}\right]$$

$$U_{eff}(t) = e^{-i\Phi/2} e^{-i\Phi\sigma_z a^\dagger a} e^{-i\Phi\sigma_z/2} \qquad \Phi = s_0 t.$$

- Apply to $|e,\alpha\rangle$

$$e^{-i\Phi a^\dagger a}|\alpha\rangle = \quad e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-i\Phi a^\dagger a}|n\rangle$$

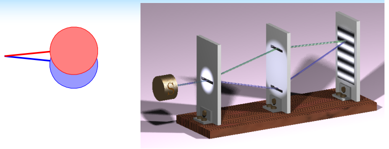
$$= e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-in\Phi}|n\rangle = |\alpha e^{-i\Phi}\rangle$$

$$\begin{aligned} |e,\alpha\rangle &\longrightarrow e^{-i\Phi}|e,\alpha e^{-i\Phi}\rangle \\ |g,\alpha\rangle &\longrightarrow |g,\alpha e^{i\Phi}\rangle \ . \end{aligned}$$

  - The atom (quantum system) controls the classical phase of the field
  - At the heart of Schrödinger cat states generation

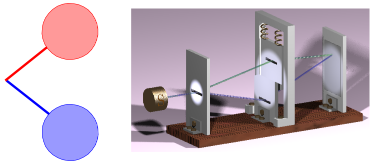
Two limiting cases

- Small phase shift (large D)
(smaller than quantum phase noise)



  - field phase almost unchanged
  - No which path information
  - Standard Ramsey fringes

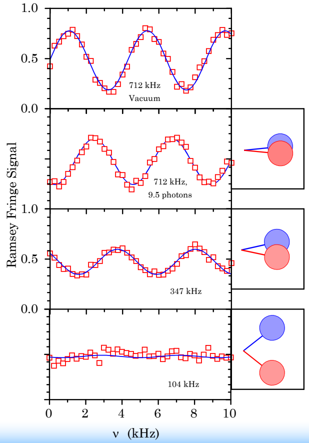
- Large phase shift (small D)
(larger than quantum phase noise)



  - Cavity fields associated to the two paths distinguishable
  - Unambiguous which path information
  - No Ramsey fringes

Fringes and field state

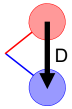
- An illustration of complementarity



Signal analysis

Fringe signal multiplied by $\left\langle \alpha e^{i\Phi} \middle| \alpha e^{-i\Phi} \right\rangle$

- Modulus $e^{-2\bar{n}\sin^2\Phi} = e^{-D^2/2}$

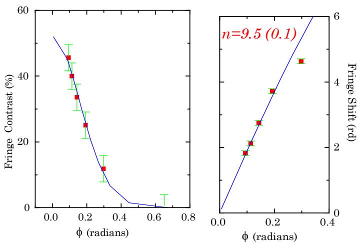


  - Contrast reduction

- Phase $2\bar{n}\sin\Phi$
  - Phase shift corresponding to cavity light shifts

Phase leads to a precise (and QND) measurement of the average photon number

Fringes contrast and phase

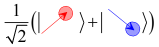


- Excellent agreement with theoretical predictions.
- Not a trivial fringes washing out effect

Calibration of the cavity field
9.5 (0.1) photons

A laboratory version of the Schrödinger cat

Field state after atomic detection



$\frac{1}{\sqrt{2}}\left(|\ \rangle + |\ \rangle\right)$

A coherent superposition of two "classical" states.

Very similar to the Schrödinger cat

Decoherence will transform this superposition into a statistical mixture

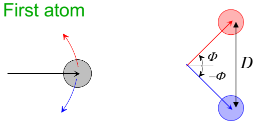
Slow relaxation: possible to study the decoherence dynamics

Decoherence caught in the act

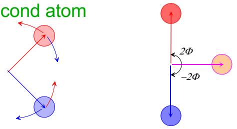
An atom to probe field coherence

Quantum interferences involving the cavity state

First atom



Second atom

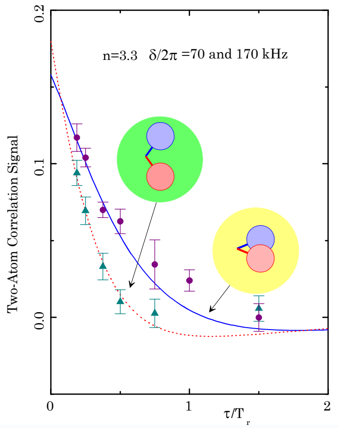


Two indistinguishable quantum paths to the same final state:

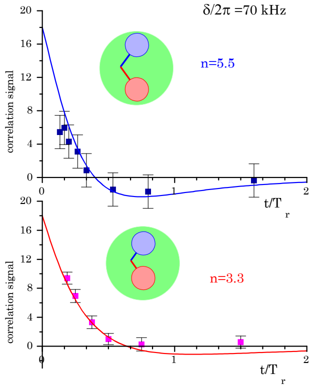
Quantum interference in a two-atom correlation signal

A decoherence study

Atomic correlation signal



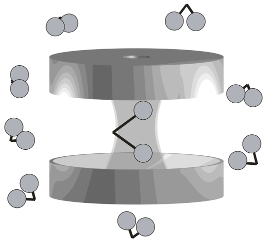
Decoherence versus size of the cat



PRL **77**, 4887 (1996)

A simple calculation of a cat's decoherence

- A cat in a cavity coupled to a bath of linear oscillators



  - Linear cavity-bath coupling: a coherent state in the cavity couples to time-dependent coherent fields in the environment modes (no cavity-environment entanglement)
  - A cat disseminates small kittens in the environment

PRL, **79**, 1964 (1997)

A simple calculation of a cat's decoherence

- Complete wavefunction at time τ:

$$\left|\alpha(\tau)e^{i\Phi}\right\rangle\prod_i\left|\beta_i(\tau)e^{i\Phi}\right\rangle+\left|\alpha(\tau)e^{-i\Phi}\right\rangle\prod_i\left|\beta_i(\tau)e^{-i\Phi}\right\rangle$$

  - Cavity state entangled with environment

- Remaining cat's coherence when tracing over the environment

$$\prod_i\left\langle\beta_i(\tau)e^{-i\Phi}\middle|\beta_i(\tau)e^{i\Phi}\right\rangle=\exp\left[-\sum_i\left|\beta_i\right|^2\left(1-e^{2i\Phi}\right)\right]$$

  - Experimental signal: 0.5x real part of this quantity

- Energy conservation

$$\sum_i\left|\beta_i(\tau)\right|^2=\overline{n}\left(1-e^{-\tau/T_r}\right)$$

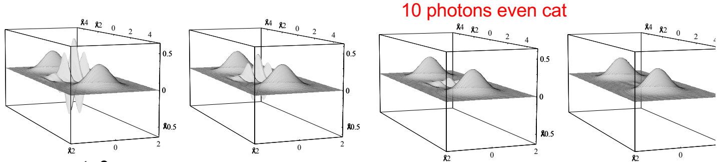
A simple calculation of a cat's decoherence

- Remaining coherence

$$\exp\left[-\overline{n}\left(1-e^{-\tau/T_r}\right)\left(1-e^{2i\Phi}\right)\right]\approx\exp\left[-2\overline{n}\tau/T_r\right]\ \text{for}\ \Phi=\pi/2$$

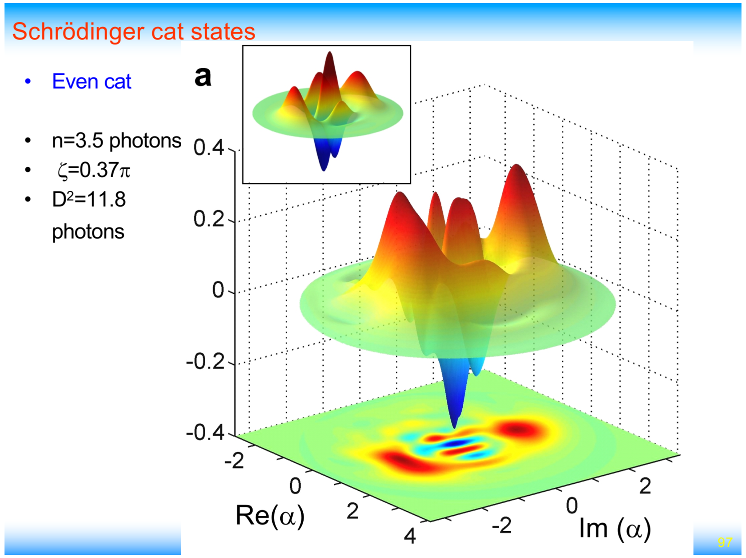
- Decoherence time scale $T_r/2\overline{n}=2T_r/D^2$ *D*: distance between cat components

10 photons even cat



t=0 t=T_c/20 t=T_c/5 t=T_c/2

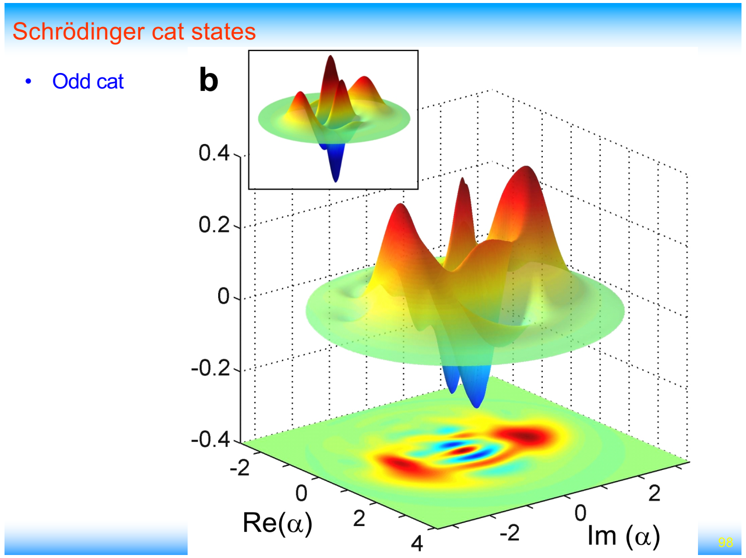
- In terms of Monte Carlo quantum trajectories
  - Cat switches parity at each photon loss
  - Parity undetermined when one photon lost on the average



Schrödinger cat states

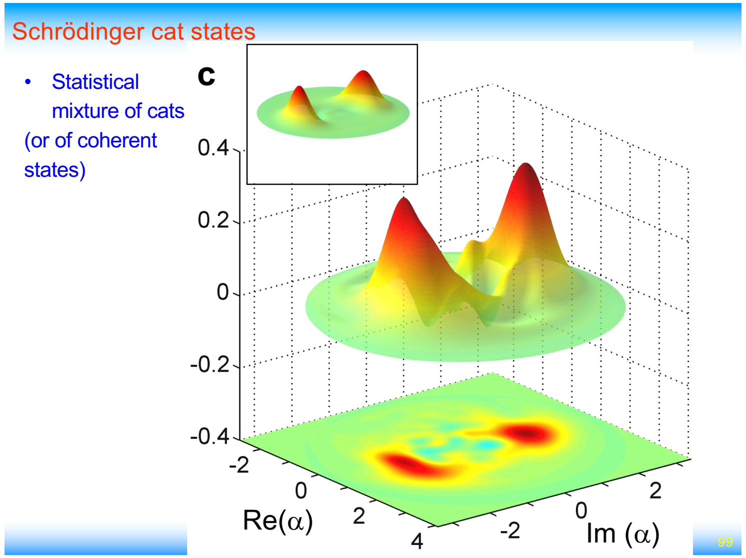
- Even cat

- n=3.5 photons
- $\zeta=0.37\pi$
- $D^2=11.8$ photons



Schrödinger cat states

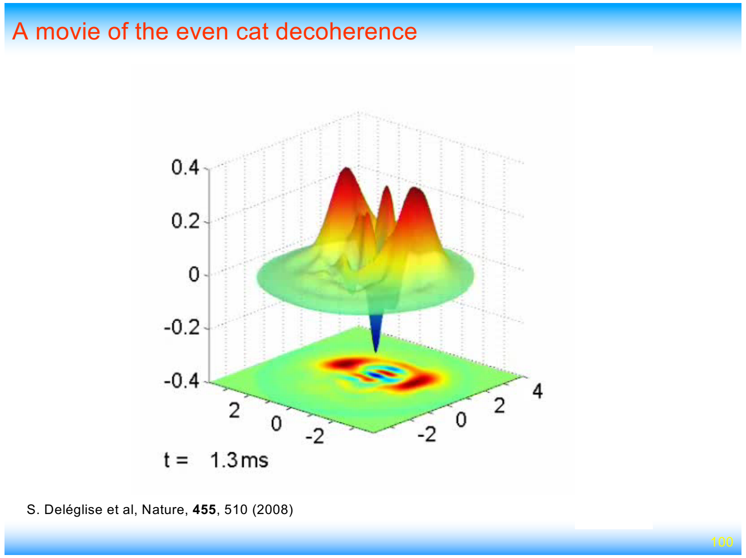
- Odd cat



Schrödinger cat states

- Statistical mixture of cats

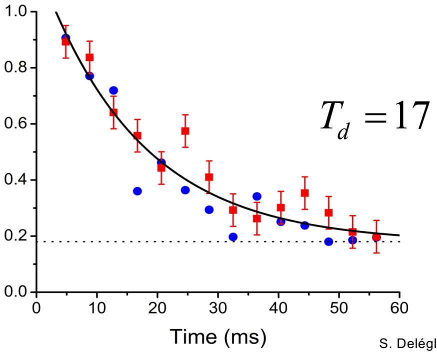
(or of coherent states)



A movie of the even cat decoherence

S. Deléglise et al, Nature, **455**, 510 (2008)

Decoherence time



$$T_d = 17 \pm 3 \text{ ms}$$

S. Deléglise et al, Nature, **455**, 510 (2008)

For similar work in circuit QED see Wang et al. PRL **103** 200404

Outline of this lecture

- Introduction
- Tools of cavity QED
- Resonant interaction
- Dispersive interaction
  - Ideal QND measurement of photon number
  - Quantum Zeno effect
  - Fock states
  - Complementarity and Schrödinger cats
  - Quantum feedback
- Perspectives

Feedback: a universal technique

- Classical feedback is present in nearly all control systems
  - A SENSOR measures the system's state
  - A CONTROLLER compares the measured quantity with a target value
  - An ACTUTATOR reacts on the system to bring it closer to the target

- Quantum feedback has the same aims for a quantum system
  - Stabilizing a quantum state against decoherence
  - Must face a fundamental difficulty:
    - measurement changes the system state

Two quantum feedback experiments

- Prepare and preserve a Fock state in the cavity
  - Target state: the photon number state n_t

- Feedback loop
  - Get information on the cavity state
    - QND quantum sensor atoms sent at 82 μs time interval
  - Estimate cavity state and distance to target
    - Fast real-time computer (ADWin Pro II)
      - A complex computation taking into account all known imperfections
    - Decide upon actuator action
  - Actuator action
    - Drives the cavity state as close as possible to the target

Two experiments

- Classical actuator
  - Actuator is a coherent source
    - Displacement of the cavity field
    - Technically simple
    - Not optimal: complex procedure to correct for single photon loss
    - Preparation and protection of Fock states up to n=4

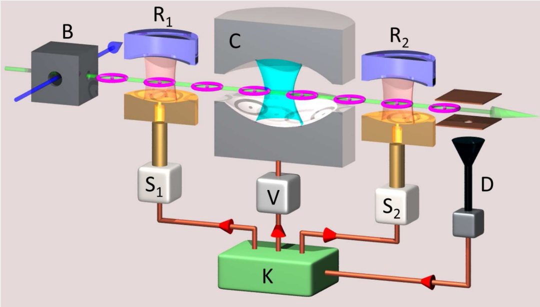
I. Dotsenko, M. Mirrahimi, M. Brune, S. Haroche, J.M. Raimond, P. Rouchon, Phys. Rev. A. 80, 013805 (2009)

C. Sayrin et al. Nature, **477**, 73 (2011)

- Quantum actuator
  - Resonant atoms used to inject/subtract photons
  - More demanding experimentally
  - Faster quantum jumps correction
  - Stabilization of Fock states up to n=7

X. Zhou *et al.*, PRL **108**, 243602 (2012)

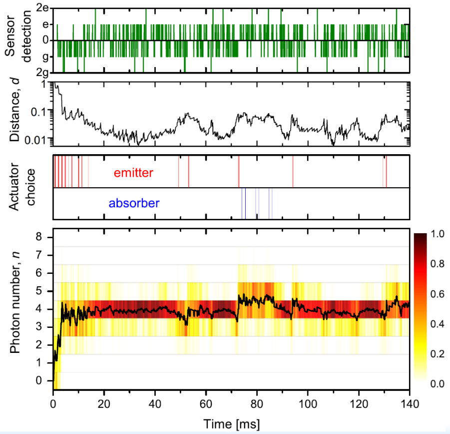
Scheme of the quantum actuator experiment



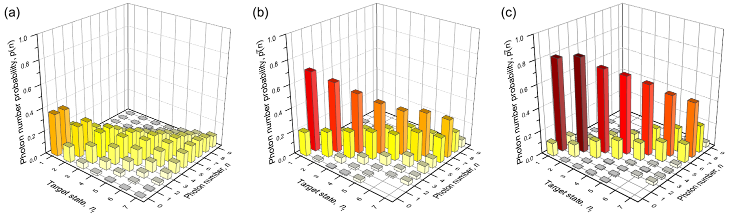
- Atomic samples
  - Sent in the cavity every 82 µs
  - Two types
    - Sensor QND samples (dispersive interaction)
    - Control samples (used by controller for feedback)
      - Absorbers, emitters or mere sensors

A single trajectory: closed loop

- Target photon number n_t=4



Feedback for high photon numbers



Reference
coherent state with
n_t photons on the average

Steady state
• stops loop at 140 ms
• independent QND estimation of average photon number distribution *P(n)*

Optimal stop
• Stops loop when *p(n_t)*>0.8
• Independent QMD estimation of *P(n)*

- Stabilization of photon numbers up to 7
- Convergence twice as fast as that of the feedback with coherent source

Using feedback to optimize QND measurement

- Send atoms one by one and use previous information to optimize information brought by next atom

- A simple scheme in an ideal setting
  - Assume n<8 (0 through 7 photons)
  - First atom sent in *g* with $\phi_0=\pi$, $\phi_r=0$
    - Detected state tells the field parity
      - Detected in *e* when empty or even photon number
      - Detected in *g* when odd photon number
    - Atom gives the Least significant bit of photon number
    - Projects the field on a parity eigenstate (cat if initial state coherent)
  - Second atom sent with $\phi_0=\pi/2$
    - Phase ϕ_r adjusted to distinguish
      - 0,4 from 2,6 if parity even
      - 1,5 from 3,7 if parity od
    - Atom gives the second bit of the photon number

Using feedback to optimize QND measurement

- A simple scheme in an ideal setting
  - Third atom sent with $\phi_0=\pi/4$
    - Ramsey phase set to remove the last ambiguity
    - Atom gives the third bit of the photon number
  - Measurement of photon number from 0 to 7 with 3 atoms
    - Instead of 110

- Straigthforward generalization
  - Measurement of photon number from 0 to N-1 with $\log_2(N)$ atoms
  - Optimum set by information theory
  - An optimal quantum digital/analog converter

- Realistic setting
  - Measure photon number from 0 to 7 with ~13 atoms (instead of 110)

Outline of this lecture

- Introduction
- Tools of cavity QED
- Resonant interaction
- Dispersive interaction
- Perspectives
  - Towards a circular state quantum simulator

Quantum simulation

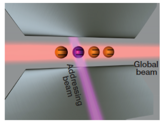
- R. Feynmann *International Journal of Theoretical Physics*, volume 21, 1982, p. 467
  - *"Nature isn't classical, dammit, and if you want to make a simulation of nature, you'd better make it quantum mechanical, and by golly it's a wonderful problem, because it doesn't look so easy".*

- The Hilbert space dimension problem
  - N spins ½: dimension 2^N. A quite rapidly growing function!
  - Explicit numerical calculations out of reach of the largest supercomputers as soon as N>42 (roughly)

- Approximate numerical methods
  - DMRG, t-DMRG, MPS,….
    - Very efficient for some questions (ground state in 1D up to N=few 100)
    - Limited for others (long-term dynamics with large entanglement, many-body localization, quenches…)

Quantum simulation

- Follow Feynman's precepts
  - Build a quantum machine to simulate the quantum
    - Realize a fully controllable/measurable system with the same dynamics as the system of interest
      - More efficient than exact classical computations for large Hilbert spaces (N>42)
- Digital quantum simulation
  - Build a full-fledged quantum computer
    - Run it to simulate the Hamiltonian of interest
      - Already implemented with ion traps
      - As difficult as a full-fledged quantum computer
        - Many qubits and many gates : "*it doesn't look so easy*"
- Analog quantum simulation



Trapped ions in 1D
Martinez et al. Nature 534 516

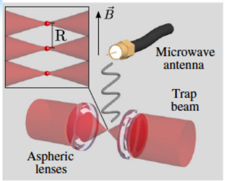
Analog quantum simulation

- A feasible approach to quantum simulation
  - Realize a N-spins system with the same dynamics as the system of interest, but which is under total control
- Main requirements
  - High-quality individual quantum systems
  - Tailorable interactions between them
  - Scalable methods for 1D-2D-3D arrangements and for initialization
  - Complete final quantum state read-out
  - Possibility to introduce a tailorable, reproducible disorder.
    - Realization *a priori* simpler than that of a full-fledged quantum computer
- One of the most promising outcomes of quantum information science
  - "*it doesn't look so easy*" but it looks feasible
  - A very active field worldwide

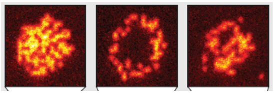
Analog quantum simulation

- Many realizations already

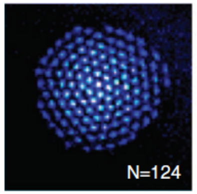
Superconducting circuits Barends et al. Nature 534,222



Rydberg atoms
Barredo et al. PRL 114 113002



Atomic lattices Zeiher et al. Nat. Phys. 12,1095



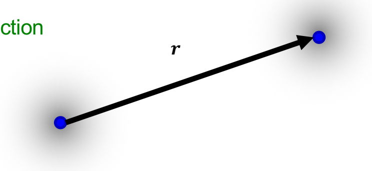
Trapped ions in 2D
Bohnet et al., Science 352 1297

And many more…

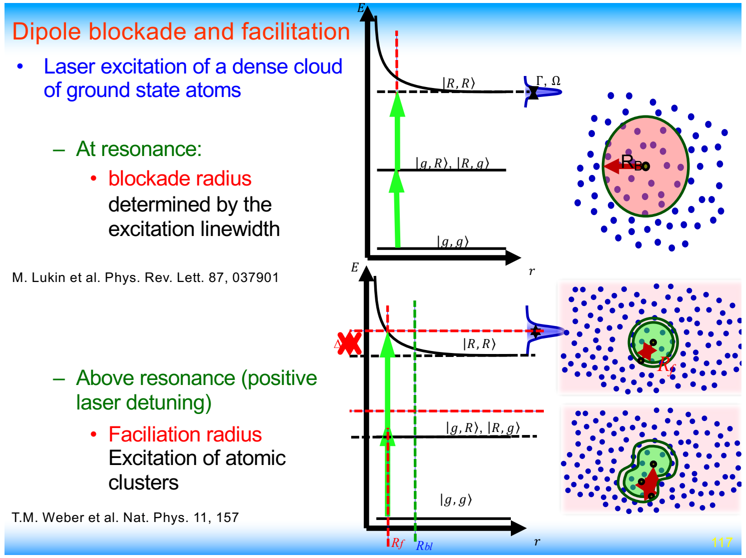
Dipole-Dipole interaction between Rydberg atoms

- A long range, strong interaction
  - Early evidence J.M. Raimond, et al J. Phys. B **14**, L655 (1981)
  - Direct measurement Béguin et al PRL **110**, 263201
  - Two 60S Rydberg levels
    - Isotropic, repulsive interaction
    - For distances > 3 µm

$$V_{dd} = \frac{C_6}{r^6}$$



- Order of magnitude
  - 8.8 MHz at 5 µm
  - To be compared with a typical 20 kHz kinetic energy in cold cloud at 1 µK



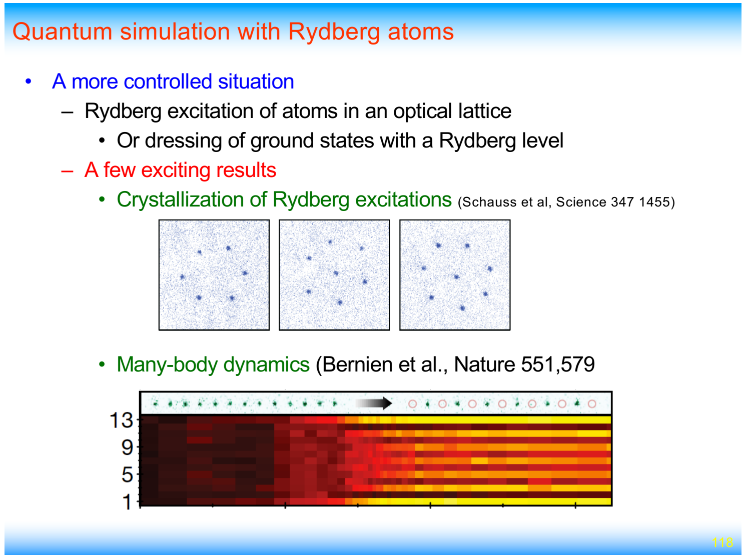
Dipole blockade and facilitation

- Laser excitation of a dense cloud of ground state atoms
  - At resonance:
    - blockade radius determined by the excitation linewidth

M. Lukin et al. Phys. Rev. Lett. 87, 037901

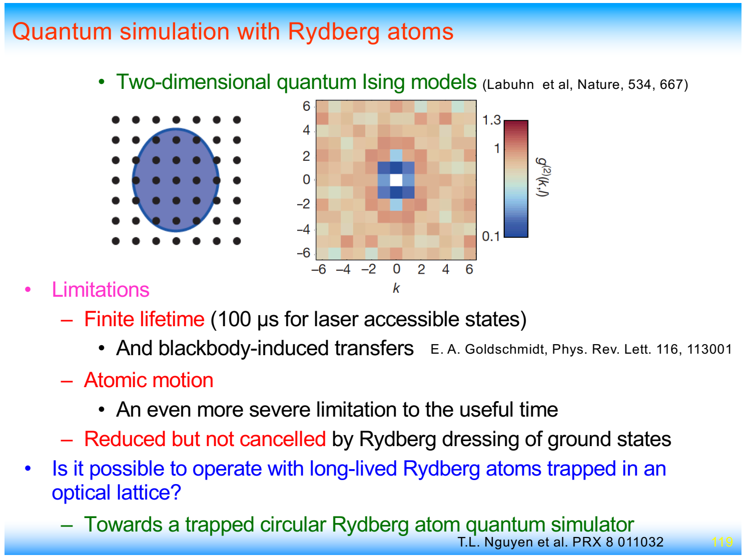
  - Above resonance (positive laser detuning)
    - Faciliation radius Excitation of atomic clusters

T.M. Weber et al. Nat. Phys. 11, 157



Quantum simulation with Rydberg atoms

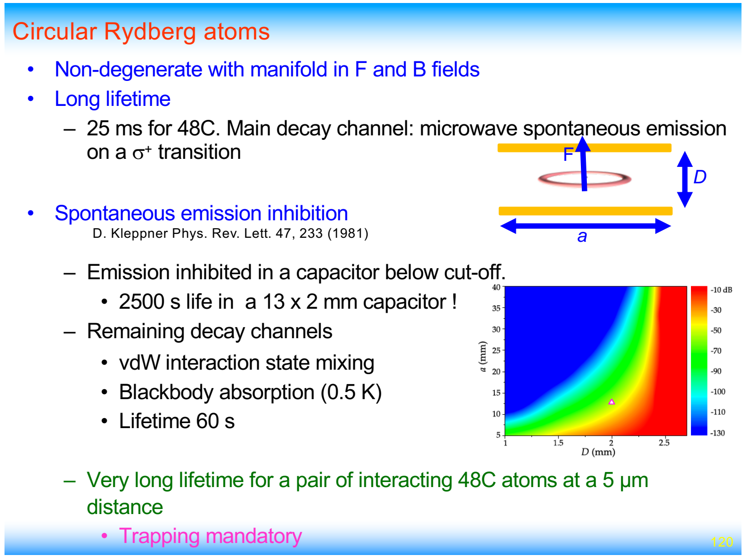
- A more controlled situation
  - Rydberg excitation of atoms in an optical lattice
    - Or dressing of ground states with a Rydberg level
  - A few exciting results
    - Crystallization of Rydberg excitations (Schauss et al, Science 347 1455)
    - Many-body dynamics (Bernien et al., Nature 551,579



Quantum simulation with Rydberg atoms

- Two-dimensional quantum Ising models (Labuhn et al, Nature, 534, 667)
- Limitations
  - Finite lifetime (100 µs for laser accessible states)
    - And blackbody-induced transfers E. A. Goldschmidt, Phys. Rev. Lett. 116, 113001
  - Atomic motion
    - An even more severe limitation to the useful time
  - Reduced but not cancelled by Rydberg dressing of ground states
- Is it possible to operate with long-lived Rydberg atoms trapped in an optical lattice?
  - Towards a trapped circular Rydberg atom quantum simulator

T.L. Nguyen et al. PRX 8 011032



Circular Rydberg atoms

- Non-degenerate with manifold in F and B fields
- Long lifetime
  - 25 ms for 48C. Main decay channel: microwave spontaneous emission on a σ^+ transition
- Spontaneous emission inhibition

D. Kleppner Phys. Rev. Lett. 47, 233 (1981)

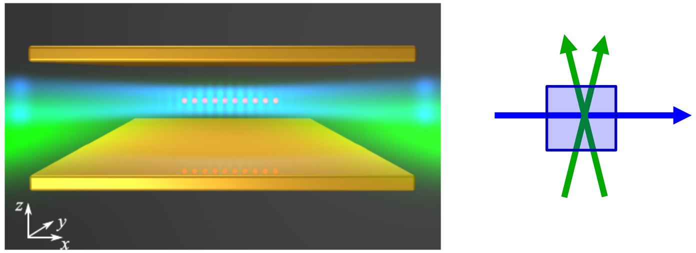
  - Emission inhibited in a capacitor below cut-off.
    - 2500 s life in a 13 x 2 mm capacitor !
  - Remaining decay channels
    - vdW interaction state mixing
    - Blackbody absorption (0.5 K)
    - Lifetime 60 s
  - Very long lifetime for a pair of interacting 48C atoms at a 5 µm distance
    - Trapping mandatory

Circular states laser trapping

- Circular states can be laser-trapped ! S. K. Dutta et al. Phys. Rev. Lett. 85, 5551
  - Ponderomotive electron energy:
    - atoms are low-field seekers
    - a large trap
      - ~10 times greater polarizability that of ground state Rubidium at 1 µm wavelength
  - Trapping almost independent of principal quantum number
    - Low trap-induced decoherence
  - Impervious to photoionization
    - severe limitation for low *l* states Saffman et al. *Phys. Rev. A* 72, 022347
- Long term trapping
  - 50 s lifetime taking into account Compton scattering and realistic vacuum conditions in a cryogenic environment
  - >1 s lifetime for a 40 atoms chain

A simple trap geometry for a 1-D lattice

- Trapping lasers at 1 µm

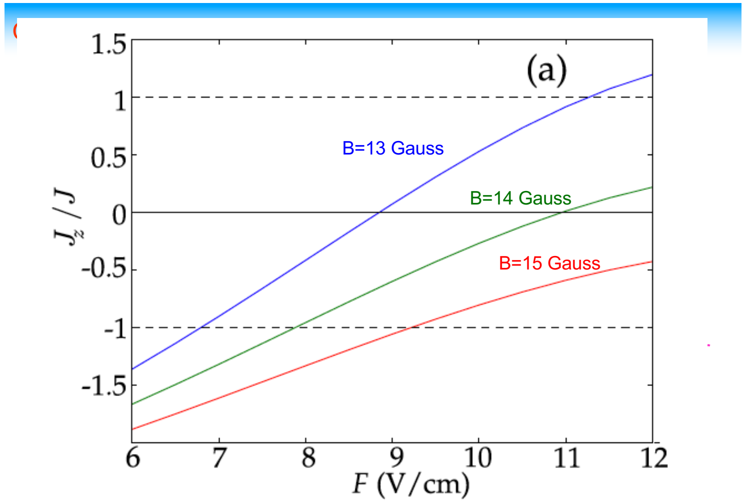


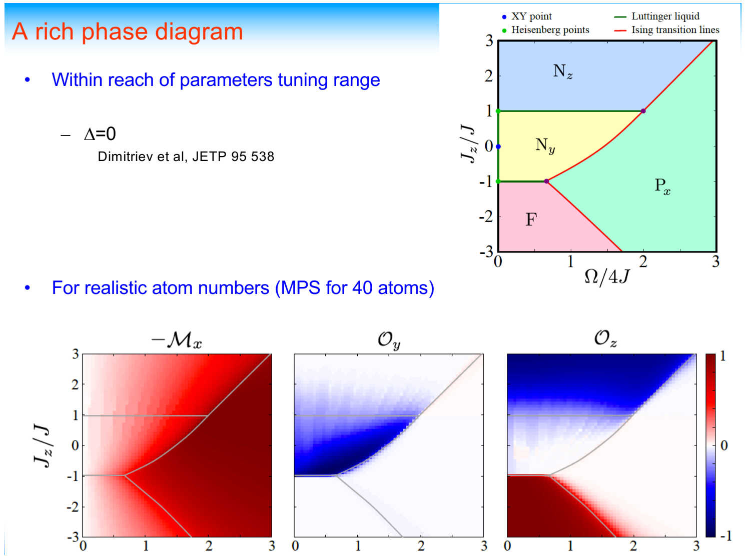
  - LG mode along Ox (transverse trap)
  - Two Gaussian beams at a small angle
    - Longitudinal lattice with an adjustable spacing
      - d= 5 to 7 µm
      - 24 kHz longitudinal oscillation frequency

Circular Rydberg interaction

- Choice of levels
  - Encode spin states on 48C and 50C
    - A repulsive van der Waal interaction (α 1/d^6) between atoms in the same levels at a distance *d*
    - A second order spin exchange interaction (48C,50C to 50C, 48C) (α 1/d^6)
- Dress the atomic transition with a near-resonant microwave
  - Rabi pulsation Ω, detuning Δ
    - Makes the ground state nontrivial
    - Can be fed in the capacitor in an evanescent mode
- Realization of the XXZ spin-1/2 chain Hamiltonian

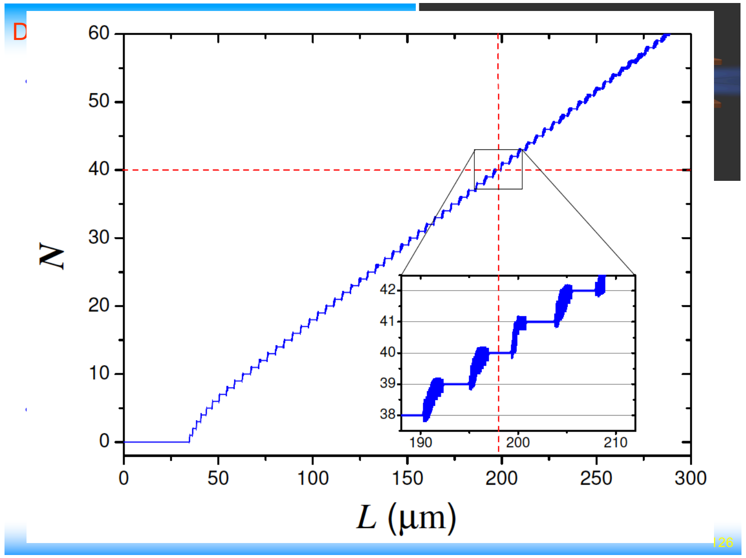
$$\frac{H}{h}=\frac{\Delta}{2}\sum_{j=1}^{N}\sigma_j^Z+\frac{\Omega}{2}\sum_{i=1}^{N}\sigma_j^X+J_Z\sum_{j=1}^{N-1}\sigma_j^Z\sigma_{j+1}^Z+J\sum_{j=1}^{N-1}(\sigma_j^X\sigma_{j+1}^X+\sigma_j^Y\sigma_{j+1}^Y)$$

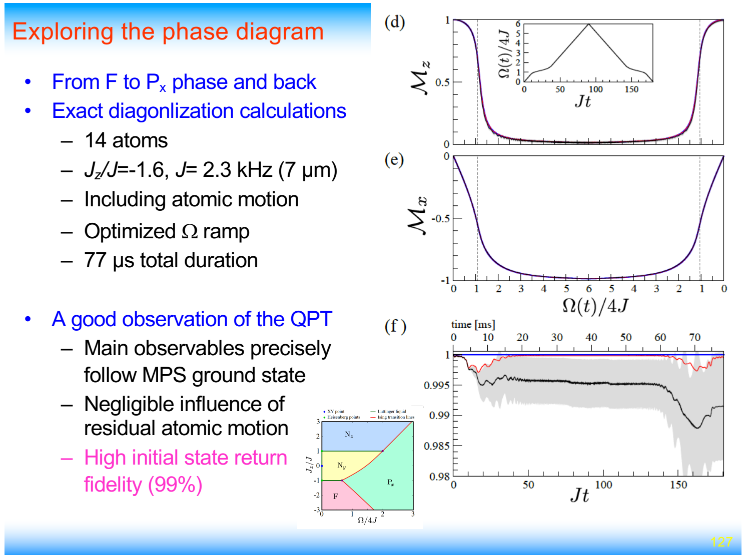




A rich phase diagram

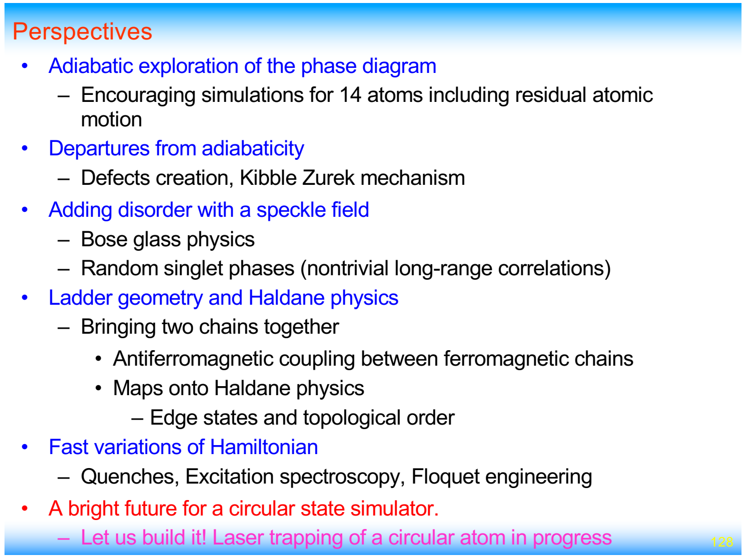
- Within reach of parameters tuning range
  - Δ=0
    Dimitriev et al, JETP 95 538
- For realistic atom numbers (MPS for 40 atoms)





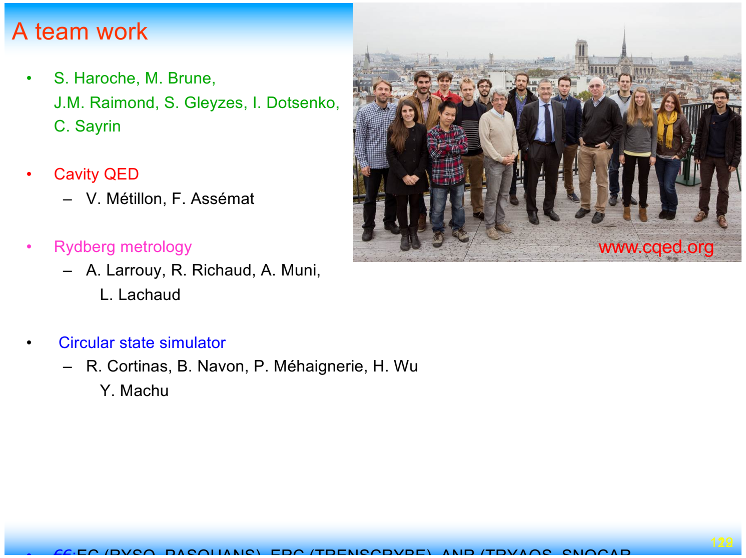
Exploring the phase diagram

- From F to P$_x$ phase and back
- Exact diagonlization calculations
  - 14 atoms
  - J_z/J=-1.6, J= 2.3 kHz (7 μm)
  - Including atomic motion
  - Optimized Ω ramp
  - 77 μs total duration
- A good observation of the QPT
  - Main observables precisely follow MPS ground state
  - Negligible influence of residual atomic motion
  - High initial state return fidelity (99%)



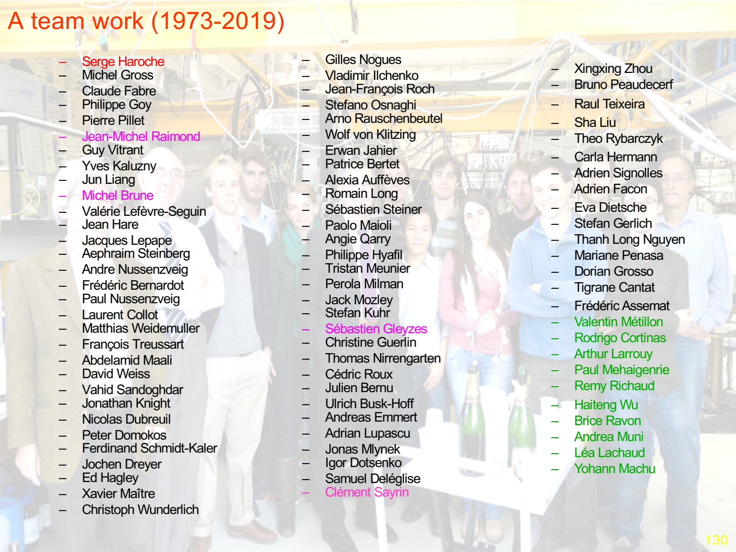
Perspectives

- Adiabatic exploration of the phase diagram
  - Encouraging simulations for 14 atoms including residual atomic motion
- Departures from adiabaticity
  - Defects creation, Kibble Zurek mechanism
- Adding disorder with a speckle field
  - Bose glass physics
  - Random singlet phases (nontrivial long-range correlations)
- Ladder geometry and Haldane physics
  - Bringing two chains together
    - Antiferromagnetic coupling between ferromagnetic chains
    - Maps onto Haldane physics
      - Edge states and topological order
- Fast variations of Hamiltonian
  - Quenches, Excitation spectroscopy, Floquet engineering
- A bright future for a circular state simulator.
  - Let us build it! Laser trapping of a circular atom in progress



A team work

- S. Haroche, M. Brune,
  J.M. Raimond, S. Gleyzes, I. Dotsenko,
  C. Sayrin
- Cavity QED
  - V. Métillon, F. Assémat
- Rydberg metrology
  - A. Larrouy, R. Richaud, A. Muni,
    L. Lachaud
- Circular state simulator
  - R. Cortinas, B. Navon, P. Méhaignerie, H. Wu
    Y. Machu



A team work (1973-2019)

- Serge Haroche
- Michel Gross
- Claude Fabre
- Philippe Goy
- Pierre Pillet
- Jean-Michel Raimond
- Guy Vitrant
- Yves Kaluzny
- Jun Liang
- Michel Brune
- Valérie Lefèvre-Seguin
- Jean Hare
- Jacques Lepape
- Aephraim Steinberg
- Andre Nussenzveig
- Frédéric Bernardot
- Paul Nussenzveig
- Laurent Collot
- Matthias Weidemuller
- François Treussart
- Abdelamid Maali
- David Weiss
- Vahid Sandoghdar
- Jonathan Knight
- Nicolas Dubreuil
- Peter Domokos
- Ferdinand Schmidt-Kaler
- Jochen Dreyer
- Ed Hagley
- Xavier Maître
- Christoph Wunderlich
- Gilles Nogues
- Vladimir Ilchenko
- Jean-François Roch
- Stefano Osnaghi
- Arno Rauschenbeutel
- Wolf von Klitzing
- Erwan Jahier
- Patrice Bertet
- Alexia Auffèves
- Romain Long
- Sébastien Steiner
- Paolo Maioli
- Angie Qarry
- Philippe Hyafil
- Tristan Meunier
- Perola Milman
- Jack Mozley
- Stefan Kuhr
- Sébastien Gleyzes
- Christine Guerlin
- Thomas Nirrengarten
- Cédric Roux
- Julien Bernu
- Ulrich Busk-Hoff
- Andreas Emmert
- Adrian Lupascu
- Jonas Mlynek
- Igor Dotsenko
- Samuel Deléglise
- Clément Sayrin
- Xingxing Zhou
- Bruno Peaudecerf
- Raul Teixeira
- Sha Liu
- Theo Rybarczyk
- Carla Hermann
- Adrien Signolles
- Adrien Facon
- Eva Dietsche
- Stefan Gerlich
- Thanh Long Nguyen
- Mariane Penasa
- Dorian Grosso
- Tigrane Cantat
- Frédéric Assemat
- Valentin Métillon
- Rodrigo Cortinas
- Arthur Larrouy
- Paul Mehaigenrie
- Remy Richaud
- Haiteng Wu
- Brice Ravon
- Andrea Muni
- Léa Lachaud
- Yohann Machu

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