Lecture 1

1. Basics: Big picture / Atom's description
2. Spin model for atom-atom interactions
   i) Handwavy, physical approach
   ii) Properties of Green's functions
   iii) Formal quantization

1. Basics

i) Applications and technology

ii) Emergent phenomena out of equilibrium

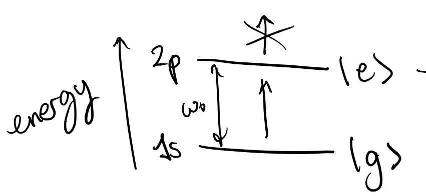
i) Applications: quantum communication, computing (gates)
metrology (atomic clocks), sensing

All this requires strong light - matter coupling. Why? Because photons do not interact with each other in vacuum. One needs matter (atoms). Matter produces:

- non-linearity / interactions (good)
- dissipation (bad)

Bulk materials not good as non-linearity is weak

Atoms are good, very non-linear due to anharmonicity of spectrum due to Coulomb potential



decay (1 photon) Γ_0

Typical energy scales

frequency: $\omega_0 \sim 300$ THz

Linewidth: $\Gamma_0 \sim 10$ MHz

[but this can change depending on atom and level. Typical atoms: Cs, Rb (alcali)]

$$Q = \frac{\omega_0}{\Gamma_0} \sim 10^7 \quad \text{very high-}Q\text{ resonators}$$

$\Rightarrow$ atoms (if not illuminated) are in ground state

- Atoms interact with light via an induced dipole moment

$$\rightsquigarrow \updownarrow^{e}_{g} \rightsquigarrow \quad \vec{p} \propto \vec{\wp}\, \sigma_{ge}$$

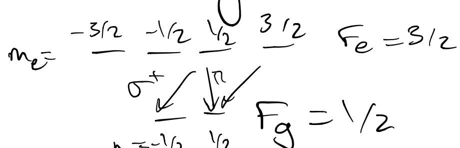
transition matrix element

$\hookrightarrow$ atomic coherence operator $|g\rangle\langle e|$

$\langle g | e\hat{r} | e \rangle$: depends on properties of the g/e wave-function

$\hookrightarrow$ selection rules (dipole-allowed transitions)

Generally atoms have hyperfine structure



Selection rules enforced by Clebsch-Gordan coeff.

Simplifications / assumptions:

- From now on we will consider then 2-level systems
- All atoms will be identical
- Goal: to describe photon-mediated inteactions btw many atoms in complex enviroments.

ii) Emergent phenomena out of Equilibrium

Dicke (1954)

Atoms radiate collectively

Superadiance / subradiance

longest lived states ←

Dicke problem is highly symmetric → easy to solve

What happens when: "crystal of atoms"

Subradiant state ~ ground state

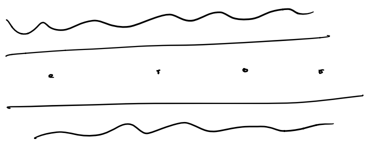
- Hilbert space atoms 2^N
- Infinite # of photon modes

Not easy ⇒ but we can integrate out photons and reduce complexity

Collective phenomena in "low-symmetry" situations:

- atom arrays
- atoms in nanophotonic structures

2. Spin model for atom-atom interactions



Let's imagine we have atoms in some given dielectric medium such as vaccum, cavity, or something more complicated.

Traditional approach: H that has both atomic and photonic degrees of freedom. Find normal mode decomposition of field and associate bosonic $a^\dagger, a$ to each mode:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{af} \text{ - interaction}$$

atom operators

$$\mathcal{H}_0 = H_{atoms} + \mathcal{H}_{field}$$

$$= \sum_{j=1}^{N} \hbar\omega_0 \sigma_{ee}^{j} + \sum_{k\mu} \hbar\omega_{k\mu} a^\dagger_{k\mu} a_{k\mu}$$

res. freq. — wave-vector — polarization

$$\sigma_{ee}^{j} = \underbrace{\sigma_{eg}^{j}}_{|e\rangle\langle g|} \underbrace{\sigma_{ge}^{j}}_{|g\rangle\langle e|} = |e\rangle\langle e|$$

photon operators

This is fine if only few modes are relevant (like high Q cavity)

However, it is complicated if there are many modes or if the expression of modes is difficult (complex dielectric structures)

We want a more general approach $\Rightarrow$ quantization using Green's functions

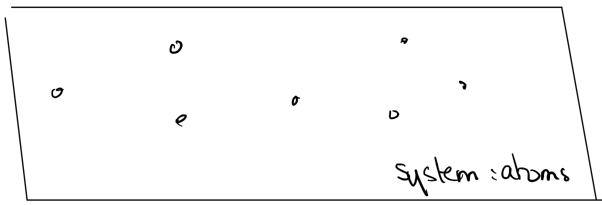
$\Rightarrow$ solve EM problem first (integrate out photonic degrees of freedom)

$\Rightarrow$ then do Quantum mechanics.

Under certain conditions (to be discussed later) one can <u>trace out</u> photonic degrees of freedom

$\Rightarrow$ Master equation for dipole - dipole interacting atoms

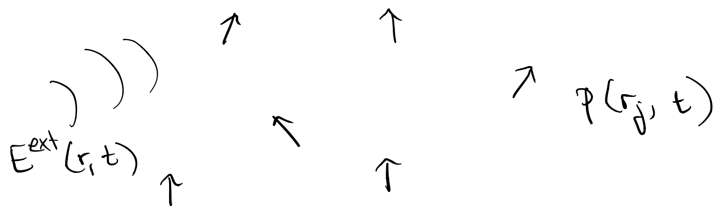
environment: photons



2.1. <u>Handwavy approach</u>: "quantizing classical dipoles"

Collection of driven dipoles in some dielectric medium

N dipoles



Monochromatic field: $E(r,t) = \underset{\overset{|||}{E(r,\omega)}}{E^{+}(r,\omega)}\, e^{-i\omega t} + \underset{\overset{|||}{E^{*}(r,\omega)}}{E^{-}(r,\omega)}\, e^{i\omega t}$

Maxwell's equations (assuming $\mu = 1$):

$$\vec{\nabla}\times\vec{E}(\vec{r},\omega) = i\omega B(\vec{r},\omega) \qquad (1)$$

$D = \epsilon_0\epsilon(r)\,E$

$$\vec{\nabla}\times\vec{H}(\vec{r},\omega) = -i\omega D(\vec{r},\omega) + \vec{j}(\vec{r},\omega) \qquad (2)$$

current density

$B = \mu_0 H$

$$\vec{\nabla}\cdot\vec{D}(r,\omega) = \rho(\vec{r},\omega)$$

charge density

$$\vec{\nabla}\cdot\vec{B}(\vec{r},\omega) = 0$$

Combining (1) and (2):

medium relative permittivity

$$\left[\vec{\nabla}\times\vec{\nabla}\times - \frac{\omega^2}{c^2}\epsilon(r)\right]\vec{E}(r,\omega) = i\omega\mu_0\vec{j}(\vec{r})$$

wave equation

We solve equation using Gf:

$$\left[\vec{\nabla}\times\vec{\nabla}\times - \frac{\omega^2}{c^2}\epsilon(r)\right]\overleftrightarrow{G}(r,r',\omega) = \mathbb{1}\delta(r-r')$$

↳ tensor

with $G(r,r',\omega) \underset{|r-r'|\to\infty}{\to} 0$

solution for point-like inhomogeneity

Fundamental solution of wave-equation

$$\Rightarrow E(\vec{r},\omega) = i\mu_0\omega\int d^3\vec{r}'\, G(r,\vec{r}',\omega)\cdot\vec{j}(\vec{r}',\omega)$$

One can check this is solution by plugging it into wave eq.

Gf is the propagator of EM field

Now current is created by point dipole:

$$\vec{j}(\vec{r},\omega) = -i\omega p\,\delta(\vec{r}-\vec{r}_0)$$

$$\Rightarrow E(\vec{r},\omega) = \mu_0\omega^2\vec{G}(\vec{r},\vec{r}_0,\omega)\cdot\vec{p}(r_0,\omega)$$

Green's function is a tensor → polarization indices

Generally, for collection of dipoles

$$\vec{E}(r,\omega) = \underbrace{\vec{E}^{ext}(r,\omega)}_{\text{drive}} + \mu_0\omega^2\sum_j \underbrace{G(r,r_j,\omega)\cdot\vec{p}(r_j,\omega)}_{\text{scattering}}$$

2.2. Properties of Green's functions:

i) $G^*(r,\omega) = G(r,-\omega)$ causality / field is real

ii) $G(r,r',\omega) = G^T(r',r,\omega)$ reciprocity

iii) $\displaystyle\frac{\omega^2}{c^2}\int ds\, \mathrm{Im}[\epsilon(s,\omega)]\, G_{\alpha\beta}(r,s,\omega)\, G^*_{\beta\gamma}(s,r',\omega) =$

$= \mathrm{Im}\, G_{\alpha\gamma}(r,r',\omega)$

Generally the Green's function is not easy to calculate:

- "trivial" for vacuum
- doable for high sym. cases (as fiber, sphere, half space)
- can be obtained numerically using Maxwell's eq. solvers (i.e. Lumerical)

Now "handwavy" Hamiltonian:

$$\mathcal{H} \sim \sum_j p(r_j, t) \cdot E(r_j, t) \sim$$

$$\sim \sum_j p(r_j, t) \cdot E^{ext}(r_j, t) \leftarrow \text{drive}$$

interaction

$$+ \mu_0 \omega_0^2 \sum_{ij} p(r_j, t) \cdot G(r_j, r_i, \omega_0) \cdot \vec{p}(r_i, t)$$

↳ resonance freq. of dipole (Markov approx)

For an atom: $\vec{p} \propto \vec{\wp}\, \hat{\sigma}_{ge}$

$$\mathcal{H}_{af} \sim \mu_0 \omega_0^2 \sum_{ij} \vec{\wp}^* \cdot G(r_i, r_j, \omega_0) \cdot \vec{\wp}\, \hat{\sigma}_{eg}^i \hat{\sigma}_{ge}^j$$

Non-Hermitian H, flip-flop interaction

Input - output eq ("hat them up"):

$$\hat{E} = \hat{E}^{ext} + \mu_0 \omega^2 \sum_j G(r, r_j, \omega_0) \cdot \vec{\wp}\, \hat{\sigma}_{ge}^j$$

Eq. for quantum field operator: identical to classical one as q. and classical fields propagate identically.

2.3. Formal approach

Following Buhmann "Dispersion forces" book

Also: Gruner, Welsch, PRA 53, 1818 (1996)

Dung, Knoll, Welsch, PRA 66, 063810 (2002)

Buhmann, Welsch, Prog. Quantum Electron. 31, 51 (2007)

- Rigorous approach valid for arbitrary dispersive / absorbing medium. We now assume it is reciprocal and isotropic.

$$\vec{D}(r,\omega) = \underbrace{\epsilon_0 \epsilon(\vec{r},\omega)\vec{E}(\vec{r},\omega)}_{\text{deterministic}} + \underbrace{\vec{P}_N(\vec{r},\omega)}_{\text{noise associated with losses}}$$

Wave-equation:

$$\left[\vec{\nabla}\times\vec{\nabla}\times - \frac{\omega^2}{c^2}\epsilon(\vec{r},\omega)\right]\vec{E}(r,\omega) = \omega^2\mu_0 \vec{P}_N(\vec{r},\omega)$$

$$\vec{E}(\vec{r},\omega) = \mu_0\omega^2 \int d\vec{r}\,'\, G(r,r',\omega)\cdot\vec{P}_N(\vec{r},\omega)$$

We introduce linear, delta-correlated operators, chosen to fulfill field commutation relations: fundamental variables

$$H = \sum_\alpha \int d\vec{r}\int_0^\infty d\omega\ \hbar\omega\, f_\alpha^\dagger(\vec{r},\omega)\, f_\alpha(r,\omega)$$

$$\left[f_\alpha(r,\omega),\, f_\beta^\dagger(r',\omega')\right] = \delta(\vec{r}-\vec{r}\,')\,\delta(\omega-\omega')\,\delta_{\alpha\beta}$$

$$\left[f_\alpha(\vec{r},\omega),\, f_\alpha(r',\omega')\right] = 0$$

and

$$\vec{P}_N = i\sqrt{\frac{\hbar\epsilon_0}{\pi}\,\mathrm{Im}\{\epsilon(r,\omega)\}}\ \vec{f}(\vec{r},\omega)$$

(pre-factor yields correct commutation relations)

Field:

Einstein convention

$$\hat{E}_\alpha(\vec{r},\omega) = \omega^2\mu_0 \int d\vec{r}\,'\, G_{\alpha\beta}(\vec{r},\vec{r}\,',\omega)\, i\sqrt{\frac{\hbar\epsilon_0}{\pi}\,\mathrm{Im}\{\epsilon(r,\omega)\}}\ \hat{f}_\beta(\vec{r},\omega)$$

$$\hat{E}_\alpha(r) = \int_0^\infty d\omega \; E_\alpha(r,\omega) + h.c.$$

$$= \underset{\text{annihilation}}{\hat{E}^+_\alpha} + \underset{\text{creation}}{\hat{E}^-_\alpha}$$

- We consider multipolar coupling and just focus on <u>dipolar</u> term.

$$\mathcal{H}_{af} = -\sum_{j=1}^{N} \underbrace{\hat{p}(r_j)}_{\vec{p}\,\sigma^j_{ge} + \vec{p}^*\sigma^j_{eg}} \cdot \underbrace{\hat{E}(\vec{r}_j)}_{\hat{E}^+ + \hat{E}^-}$$

retaining non-rotating terms

We want to <u>integrate out photonic degrees of freedom</u> (photons become bath). Once you trace $\Rightarrow$ <u>irreversible dynamics</u> (needs density matrix).

$$\rho = \mathrm{Tr}_{field}\{\rho_{af}\}$$

$$i\hbar\dot{\rho}_{af} = [\underset{(\text{int pict})}{\mathcal{H}_{af}(t)}, \rho_{af}]$$

$$\Rightarrow \rho_{af}(t) = \rho_{af}(0) - \frac{i}{\hbar}\int_0^t [\mathcal{H}(t'), \rho_{af}(t')]\, dt'$$

Born approx: $\rho_{af}(t) = \rho(t) \otimes |0\rangle\langle 0|$. Substituting again and tracing, and making Markov's approx:

$$\rho = -\frac{1}{\hbar^2}\mathrm{Tr}_{field}\int_0^\infty d\tau \left[\mathcal{H}_{af}(t), \left[H_{af}(t-\tau), \rho(t)\otimes|0\rangle\langle 0|\right]\right]$$

only depends on atomic degrees of freedom

in interaction picture

Born-Markov

Born-Markov

- perturbative (2^{nd} order)
- reservoir has broad bandwidth (short memory)

we drop oscillating terms suc as $\omega_0+\omega_K$

And use $\displaystyle\int_0^\infty dz\, e^{-i(\omega_k \mp \omega_0)z} = \pi\delta(\omega_0\mp\omega_k) \pm i\mathcal{P}\frac{1}{\omega_0\mp\omega_k}$

Im $G(\omega_0)$ Re $G(\omega_0)$

We get terms like

$$\sigma_{eg}^k\sigma_{ge}^j,\quad \cancel{\sigma_{eg}^j\sigma_{eg}^k},\quad \cancel{\sigma_{ge}^j\sigma_{ge}^k},\quad \sigma_{ge}^j\sigma_{eg}^k$$

We do RWA and kill fast-rotating terms (beware, do not do it at H-level).

↳ problems with Casimir terms

and find <u>spin-model</u>

$$\dot\rho = -\frac{i}{\hbar}[\mathcal{H},\rho] + \mathcal{L}[\rho]$$

Casimir

coherent → $$\mathcal{H} = \hbar\omega_0\sum_j \sigma_{ee}^j + \hbar\sum_{ij} J_{ij}\,\sigma_{eg}^i\sigma_{ge}^j + \mathcal{H}_c$$

$$\mathcal{L}[\rho] = \sum_{ij}\frac{\Gamma_{ij}}{2}\left(2\sigma_{ge}^i\rho\,\sigma_{eg}^j - \sigma_{eg}^i\sigma_{ge}^j\rho - \rho\,\sigma_{eg}^i\sigma_{ge}^j\right)$$

dissipative (Lindblad)

Markov

$$J_{ij} = -\frac{\mu_0\omega_0^2}{\hbar}\,\vec{p}^{*}\cdot \mathrm{Re}\,G(r_i,r_j,\omega_0)\cdot\vec{p}$$

$$\Gamma_{ij} = \frac{2\mu_0\omega_0^2}{\hbar}\,\vec{p}^{*}\cdot \mathrm{Im}\,G(r_i,r_j,\omega_0)\cdot\vec{p}$$

$$\mathcal{H}_c = -\frac{\mu_0}{\pi}\sum_{ij}\sigma_{eg}^i\sigma_{ge}^j\int_0^\infty du\,\frac{u^2\omega_0}{u^2+\omega_0^2}\,\vec{p}^{*}\cdot\mathrm{Re}\{G(r_i,r_j,iu)\}\cdot\vec{p}$$
$$+\frac{\mu_0}{\pi}\sum_{ij}\sigma_{ge}^i\sigma_{eg}^j\int_0^\infty du\,\frac{u^2\omega_0}{u^2+\omega_0^2}\,\vec{p}^{*}\cdot\mathrm{Re}\{G(r_i,r_j,iu)\}\cdot\vec{p}$$

If $i\neq j \Rightarrow \mathcal{H}_c = 0$.

XY model with long-range interactions, open

To avoid dealing with Lindblad operators and density matrix we define:

$$\mathcal{H}_{eff} = -\mu_0\omega_0^2 \sum_{ij} \mathbf{p}^* \cdot G(r_i, r_j, \omega_0) \cdot \mathbf{p}\, \sigma_{eg}^i \sigma_{ge}^j$$

- non-Hermitian, but conserves number of excit. ↳ non-unitary dynamics
- to evolve apply jumps

Non-Hermitian QM: complex symmetric matrix
non-normal modes

Input - output:

$$\hat{E} = E_{in} + \mu_0\omega_0^2 \sum_j G(r, r_j, \omega_0) \cdot \mathbf{p}\, \hat{\sigma}_{ge}^j$$

- classical ≡ quantum