Lectures on quantum gases

Lecture 3

Elastic Collision Phenomena in Quantum Gases

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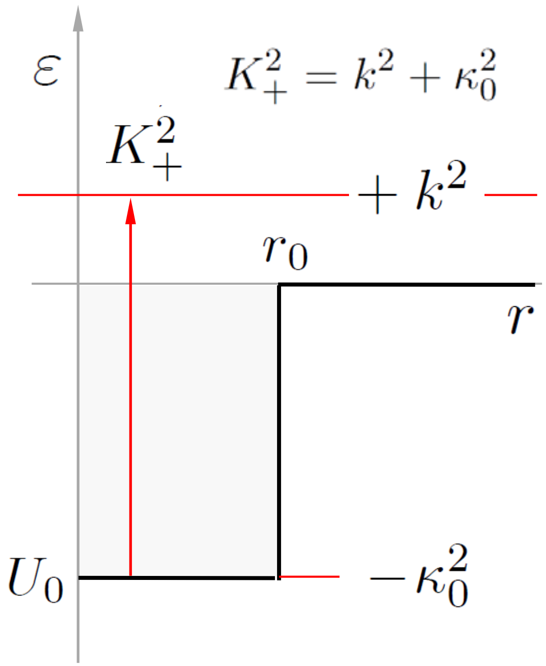
Where do we stand?

1. We identified the phase shift as the central quantity for scattering
2. We determined the scattering amplitude (4 expressions for f_l)
3. We derived the differential and total cross sections
4. We identified the role of quantum statistics
5. We studied the phase shift for hard spheres
6. We studied the phase shift for spherical square wells
7. We defined the scattering length
8. We studied its dependence on the well parameter

s-wave scattering length *a*



$$\eta_0(k) = -kr_0 + \arctan\frac{k}{K_+ \cot K_+ r_0}$$

Introduce *effective hard sphere diameter* $a(k)$

$$\eta_0(k) \equiv -ka(k)$$

We changed parameters:

$$\{A_0, B_0\} \rightarrow \{A, \eta_0\} \rightarrow \{A, a\}$$

$$\boxed{a(k) = r_0 - \frac{1}{k}\arctan\frac{k}{K_+ \cot K_+ r_0}}$$

Define scattering length: $a \equiv \lim_{k\to 0} a(k) = -\lim_{k\to 0} \eta_0(k)/k$

$$K_+^2 = k^2 + \kappa_0^2 \xrightarrow[k\to 0]{} K_+ \to \kappa_0$$

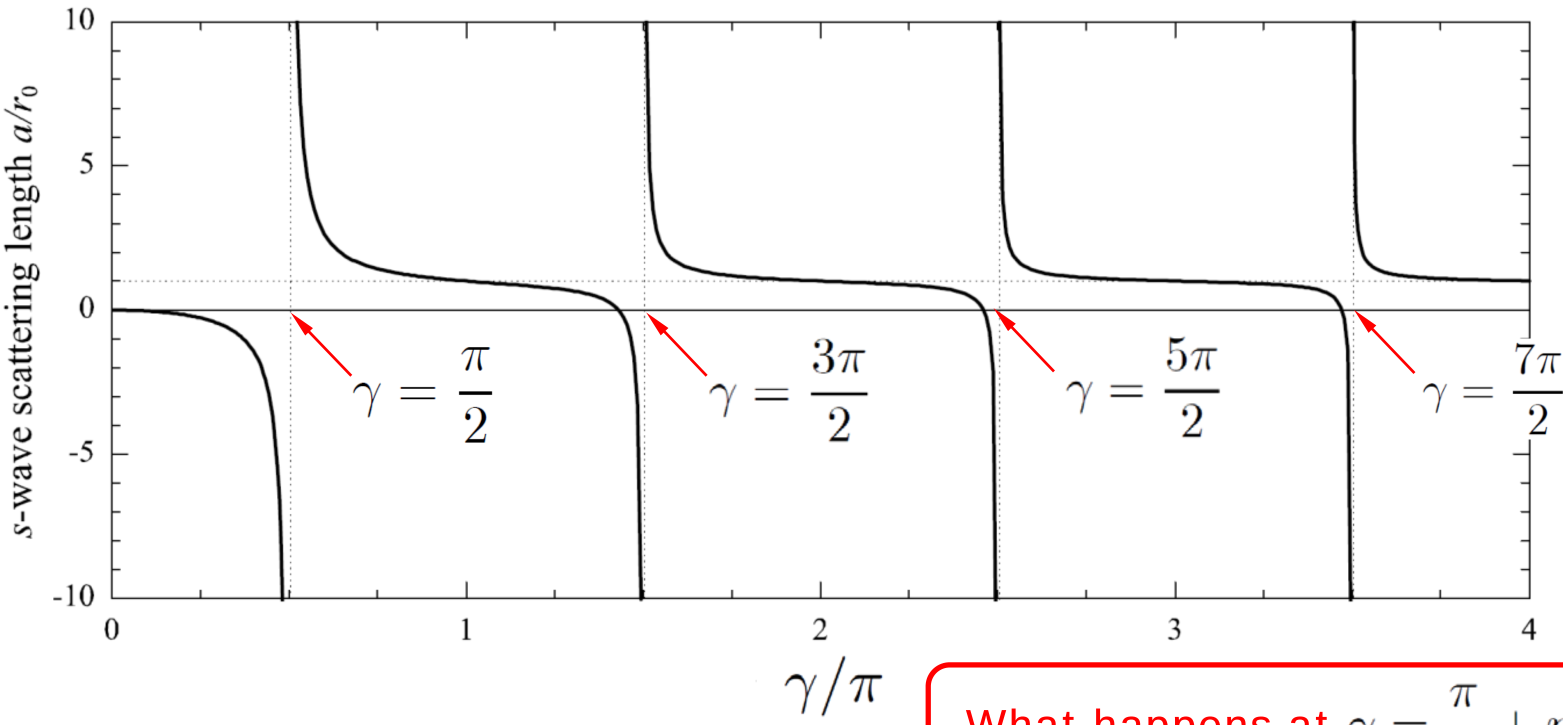
$$\longrightarrow \quad a = r_0 - \frac{r_0}{\underbrace{\kappa_0 r_0}_{\gamma} \cot \underbrace{\kappa_0 r_0}_{\gamma}} \quad \longrightarrow \quad \boxed{\kappa_0 \cot \kappa_0 r_0 = \frac{1}{r_0 - a}}$$

Define well parameter: $\gamma \equiv \kappa_0 r_0$ $\longrightarrow$ $\boxed{a = r_0\left(1 - \tan\gamma/\gamma\right)}$

s-wave scattering length a

well parameter

$$\boxed{a = r_0\left(1 - \tan\gamma/\gamma\right)} \qquad \gamma \equiv \kappa_0 r_0$$



What happens at $\gamma = \dfrac{\pi}{2} + n\pi$?

collisional significance of scattering length

$$f_l = \frac{1}{2ik}(e^{2i\eta_l} - 1) \quad \text{(a)}$$

$$= k^{-1} e^{i\eta_l} \sin\eta_l \quad \text{(b)} \longrightarrow f_l = \frac{1}{k} e^{i\eta_l} \sin\eta_l \xrightarrow{\eta_0 = -ka} \boxed{f_0 \underset{k\to 0}{\simeq} -a}$$

$$= \frac{1}{k\cot\eta_l - ik} \quad \text{(c)}$$

$$= k^{-1}\left(\sin\eta_l \cos\eta_l + i\sin^2\eta_l\right) \quad \text{(d)}$$

Bosons (<u>even</u> partial waves): s-wave scattering

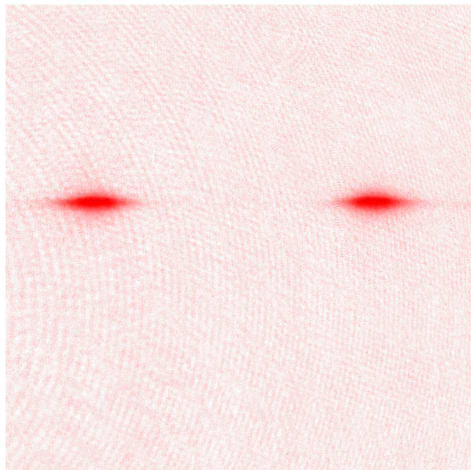
$$\sigma = \frac{8\pi}{k^2} \sum_{l=\text{even}} (2l+1)\sin^2\eta_l \longrightarrow \sigma_0 = \frac{8\pi}{k^2}\sin^2\eta_0 \xrightarrow{\eta_0 = -ka} \sigma_0 = \frac{8\pi}{k^2}\sin^2 ka$$

$$\big\downarrow k \to 0$$

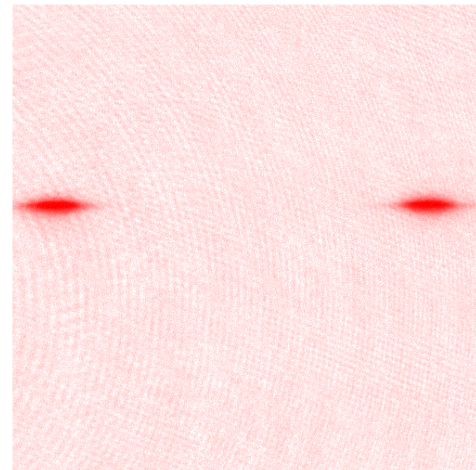
$$\boxed{\sigma_0 = 8\pi a^2}$$

collisions of ultracold atoms

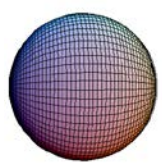
E_c/k_B=138 μK



E_c/k_B=1230 μK



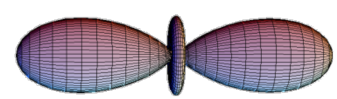
~1mm



$^{87}\mathrm{Rb}\ \left|F=2, m_F=2\right\rangle$

$Y_0^0(q)$

s-wave



$Y_2^0(q)$

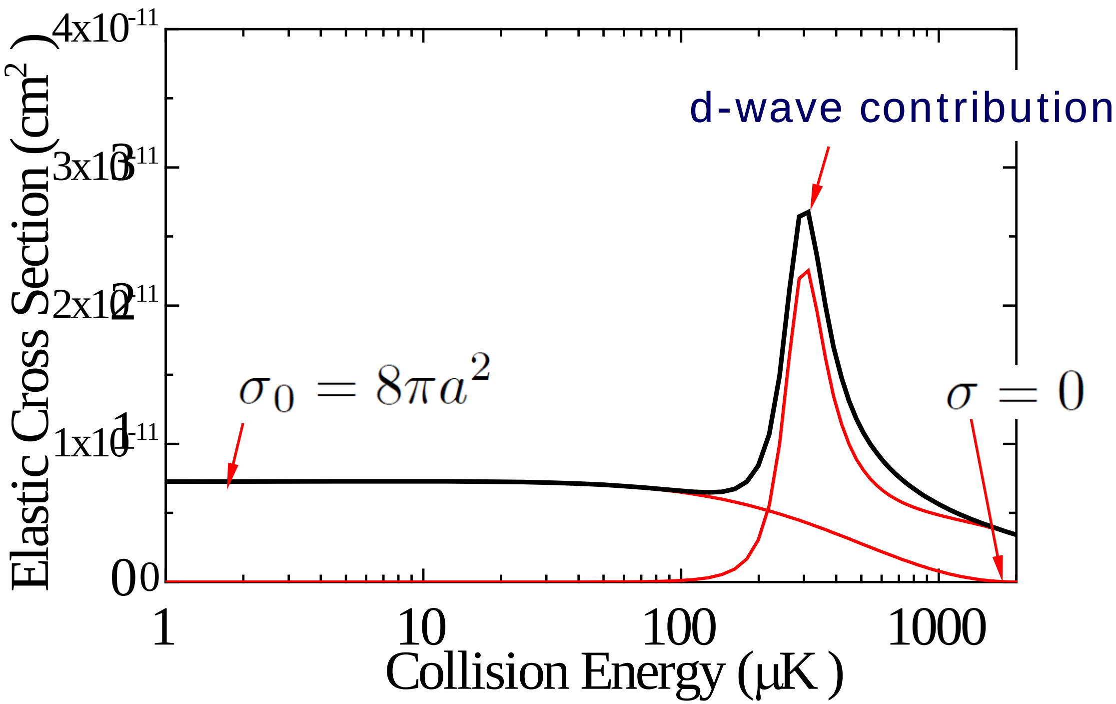
d-wave

elastic cross section for identical bosons

$$\sigma = \frac{8\pi}{k^2} \sum_{l=\text{even}} (2l+1)\sin^2\eta_l \longrightarrow \sigma_0 = \frac{8\pi}{k^2}\sin^2\eta_0 \xrightarrow{\eta_0 = -ka} \sigma_0 = \frac{8\pi}{k^2}\sin^2 ka$$

$\eta_0 \to \pi$: $\boxed{\sigma = 0}$ → First Ramsauer minimum

$k \to 0$: $\boxed{\sigma_0 = 8\pi a^2}$



no p-wave !

divergence of the scattering length

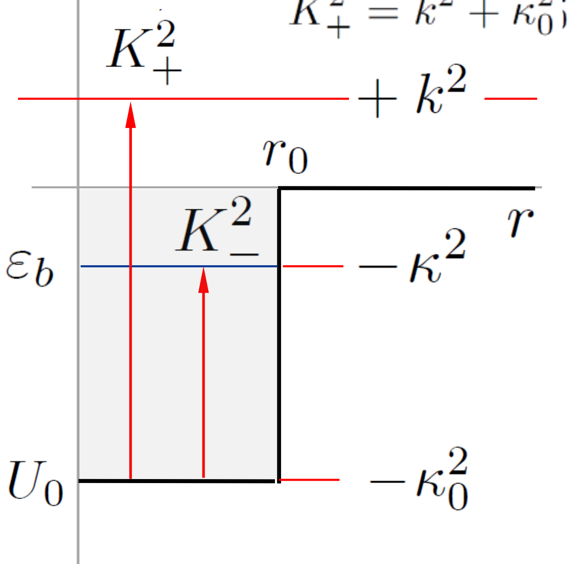
What is happening?

universal behavior near threshold

spherical square well for $l = 0$ and $\varepsilon < 0$

spherical square well for $l = 0$ and $\varepsilon < 0$

$$\chi_0'' + \underbrace{\left[k^2 - U(r)\right]}_{K_+^2 = k^2 + \kappa_0^2} \chi_0 = 0$$



$r > r_0 \quad U_0(r) = 0 \qquad \chi_0'' - \kappa^2 \chi_0 = 0$

normalization

$$\chi_0 = A e^{-\kappa r} \qquad (\kappa > 0)$$
$$\chi_0' = -\kappa A e^{-\kappa r}$$

$r \le r_0 \quad U_0(r) = -\kappa_0^2 \qquad \chi_0'' + K_-^2 \chi_0 = 0$

$$\chi_0 = A' \sin(K_- r + \cancel{\eta_0})$$
$$\chi_0' = K_- A' \cos(K_- r)$$

boundary condition: $\chi_0(r)$ and $\chi_0'(r)$ continuous at $r = r_0$

$$\chi_0'/\chi_0|_{r=r_0} = \underset{r \le r_0}{K_- \cot K_- r_0} = \underset{r > r_0}{-\kappa}$$

Bethe-Peierls boundary condition

$$K_-^2 = \kappa_0^2 - \kappa^2 \xrightarrow[\kappa \to 0]{} K_- \to \kappa_0 \longrightarrow \chi_0'/\chi_0|_{r=r_0} = \kappa_0 \cot \underbrace{\kappa_0 r_0} = 0 \text{ for } \kappa \to 0$$

Conclusion: next bound state appears for $\gamma = \dfrac{\pi}{2} + n\pi$

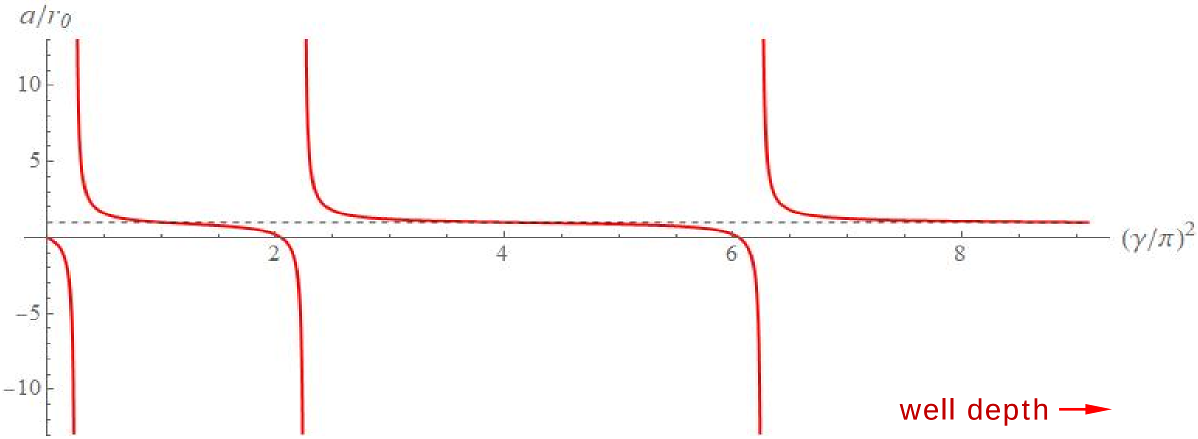
binding energy versus well parameter

$$\chi_0'/\chi_0|_{r=r_0} = K_- \cot K_- r_0 = -\kappa$$

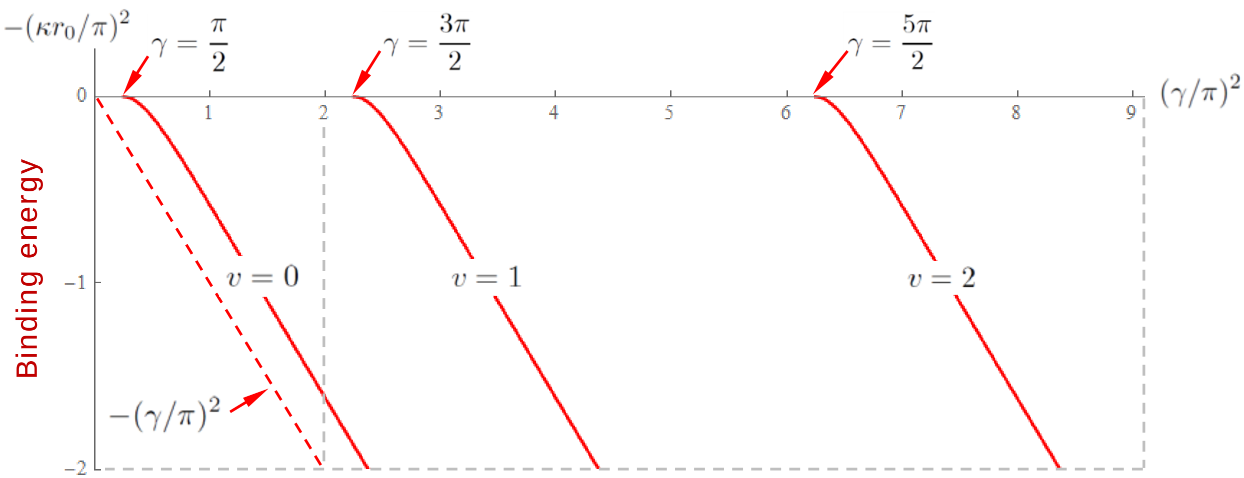
$$K_-^2 = -\kappa^2 + \kappa_0^2$$

well parameter: $\gamma \equiv \kappa_0 r_0$

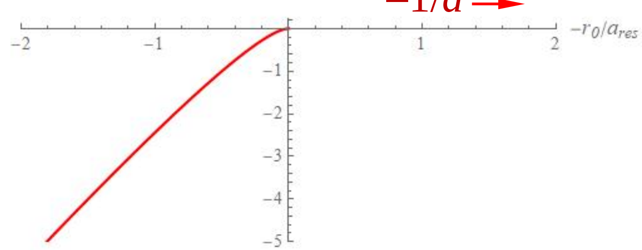
$$\boxed{E_b = -\frac{\hbar^2}{2\mu}\frac{1}{a^2}}$$



well depth →

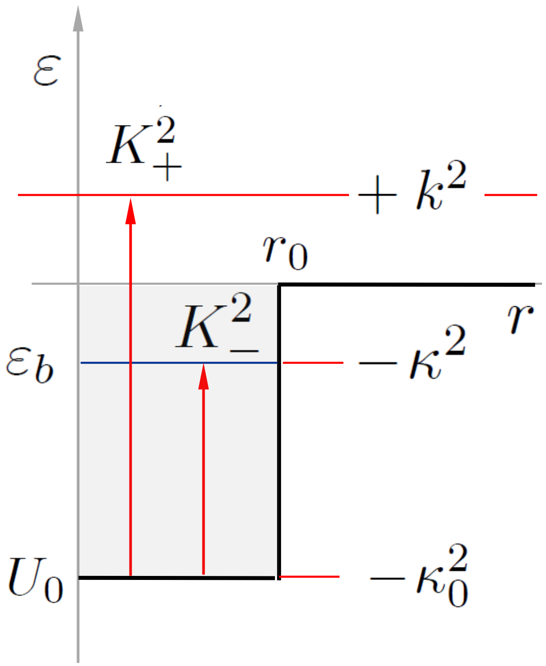


$-1/a$ →



Binding energy ↓

universal behavior near threshold



Continuum state near threshold:

$$\varepsilon > 0 \quad (k \to 0) \quad \kappa_0 \cot \kappa_0 r_0 = \frac{1}{r_0 - a} \simeq -\frac{1}{a}$$

Bound state near threshold:

$$\varepsilon < 0 \quad (\kappa \to 0) \quad \kappa_0 \cot \kappa_0 r_0 \simeq -\kappa \quad (\kappa > 0)$$

$$\longrightarrow \boxed{a = \frac{1}{\kappa}} \longrightarrow \boxed{a > 0 \quad \text{for} \quad \kappa \to 0}$$

Universal dependence of binding energy on scattering length:

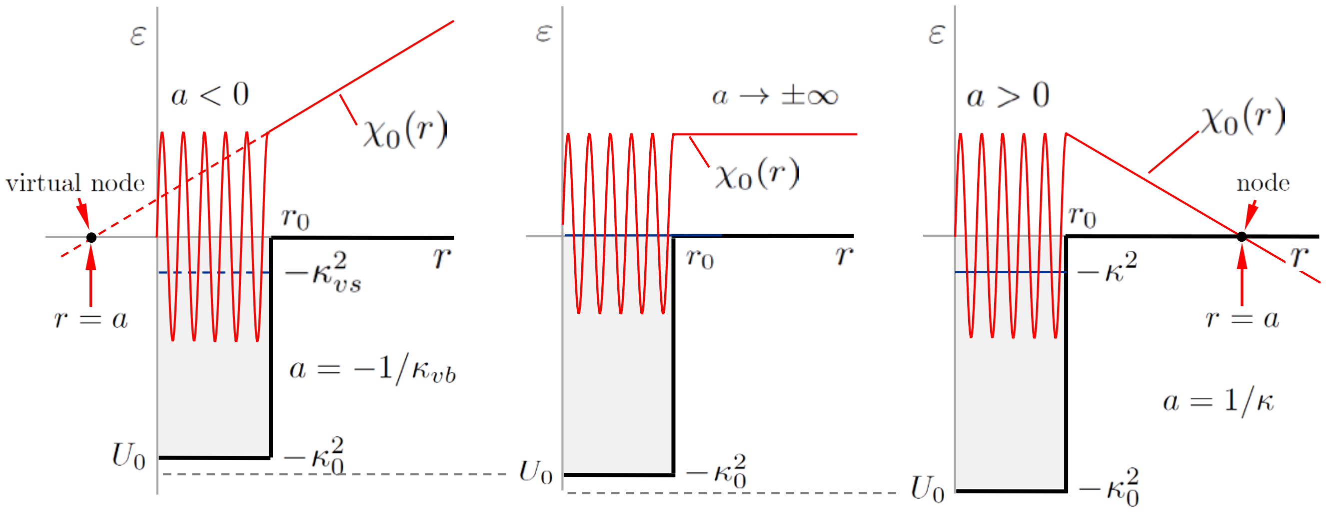
$$\varepsilon = -\kappa^2 = -\frac{1}{a^2} \longrightarrow \boxed{E_b = -\frac{\hbar^2}{2\mu}\frac{1}{a^2}}$$

scattering length a

Continuum states ($\varepsilon > 0$):

$$\eta_0 = -ka$$

$$\chi_0(r) = \sin(kr + \eta_0) = \sin[k(r-a)] \underset{k\to 0}{\simeq} k(r-a)$$



$$\varepsilon_{vs} = -\kappa^2 = -\kappa_{vs}^2 \qquad \varepsilon = 0 \qquad \varepsilon = -\kappa^2$$

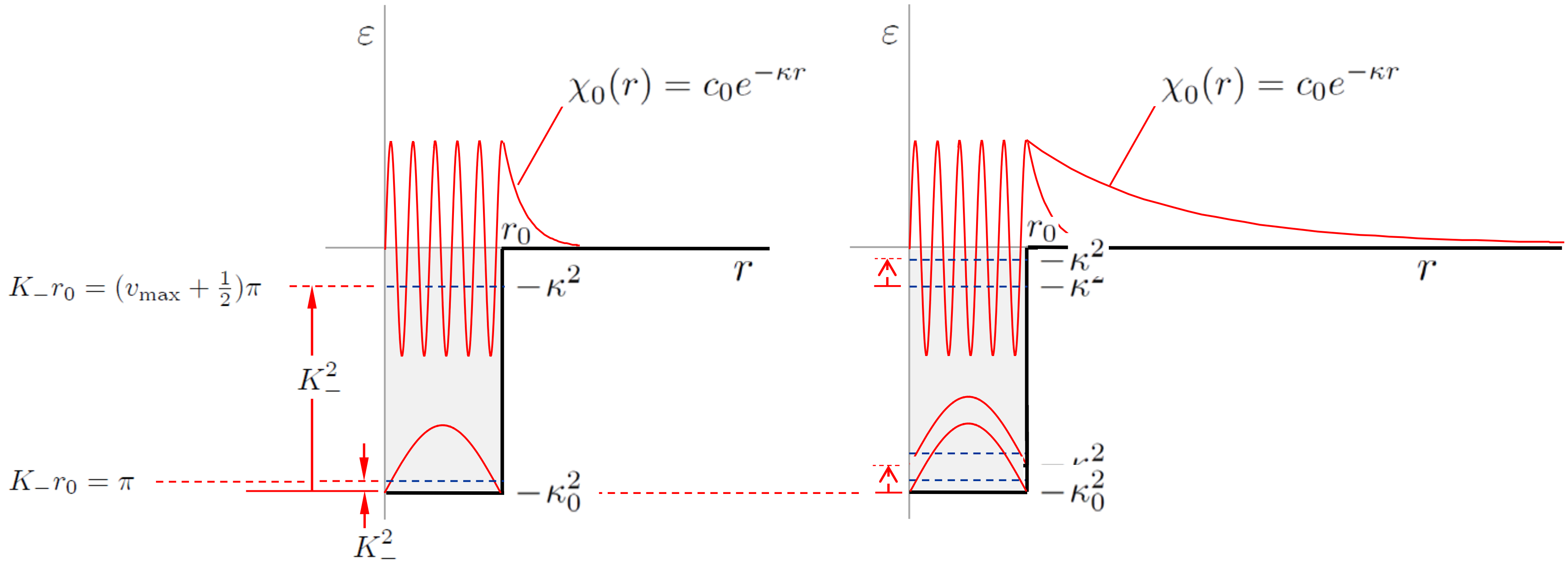
$$\kappa = -\kappa_{vs}$$

halo states

Bound states ($\varepsilon < 0$):

$$\left.\begin{array}{r}\chi_0 = Ae^{-\kappa r} \\ \kappa r_0 \ll 1\end{array}\right] \rightarrow$$ universal bound states: halo states

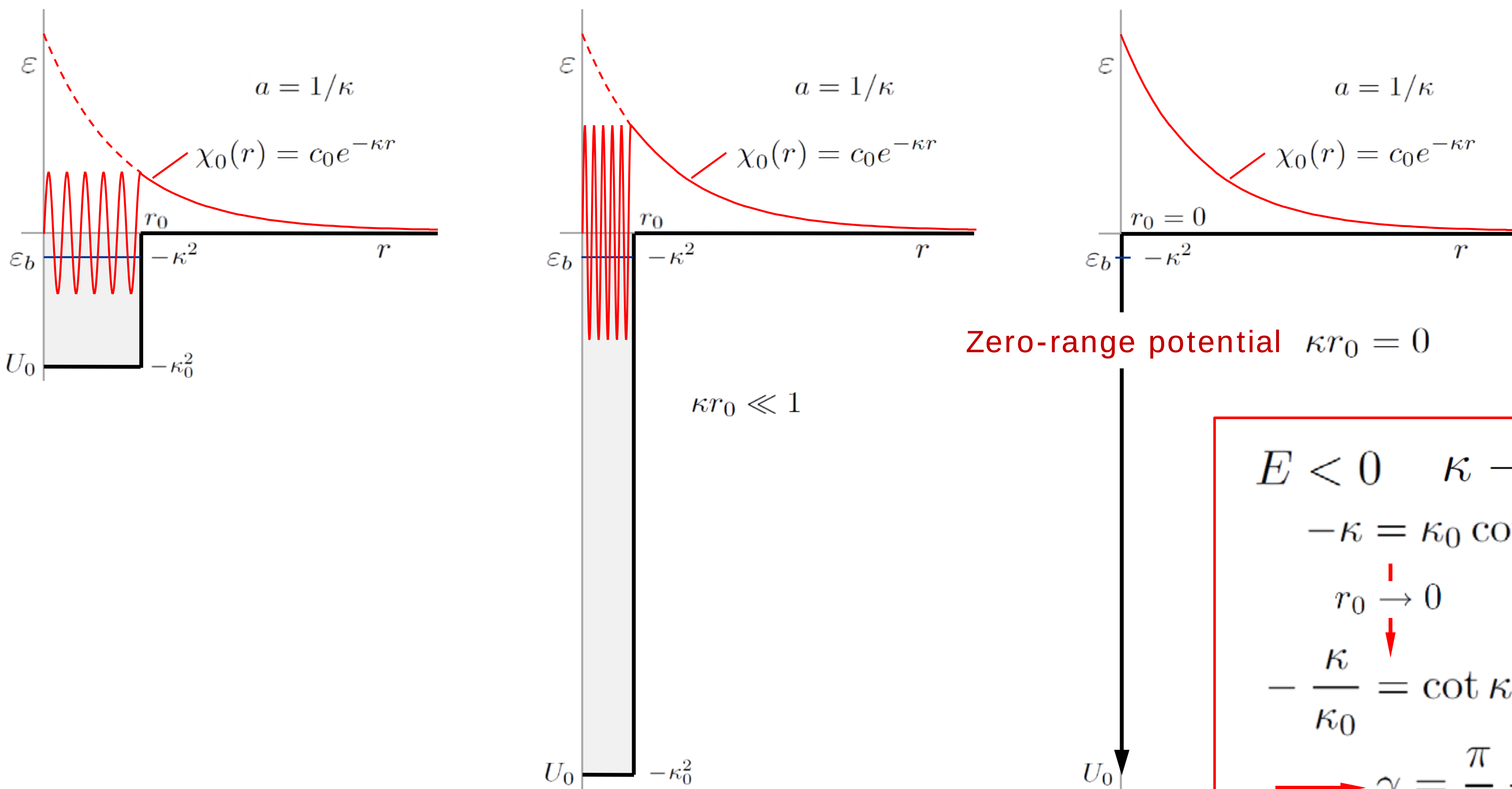
strange molecule: large probability to find the atoms *outside* the classical turning point



halo states

Bound states ($\varepsilon < 0$):

$$\left.\begin{array}{r} \chi_0 = Ae^{-\kappa r} \\ \kappa r_0 \ll 1 \end{array}\right] \longrightarrow$$ universal bound states: halo states



$$E < 0 \qquad \kappa \rightarrow 0$$

$$-\kappa = \kappa_0 \cot \kappa_0 r_0$$

$$r_0 \rightarrow 0$$

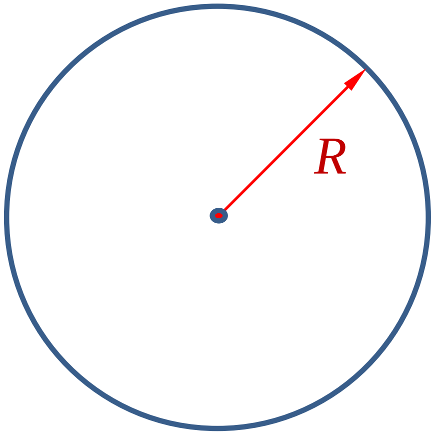
$$-\frac{\kappa}{\kappa_0} = \cot \kappa_0 r_0 \rightarrow 0$$

$$\longrightarrow \gamma = \frac{\pi}{2} + n\pi$$

significance of *s*-wave scattering length

We distinguish between: (a) thermodynamic significance
(b) collisional significance

Thermodynamic significance



particle spherical (macroscopic) box:

(a) without interaction

$$R_0^{(0)}(k_n, r) = \sqrt{\frac{2}{R}} \frac{\sin[k_n r]}{r} \qquad k_n = n\frac{\pi}{R}$$

(b) with interaction

$$\eta_0 = -k_n' a$$

$$R_0(k_n', r) \underset{r>r_0}{=} \sqrt{\frac{2}{R}} \frac{\sin[k_n' r + \eta_0(k_n')]}{r} \qquad k_n' = n\frac{\pi}{R-a}$$

$$\boxed{\delta E = \frac{\hbar^2}{2m}\left(k_n'^2 - k_n^2\right) \simeq n^2 \frac{\hbar^2}{m}\frac{\pi^2}{R^3} a}$$

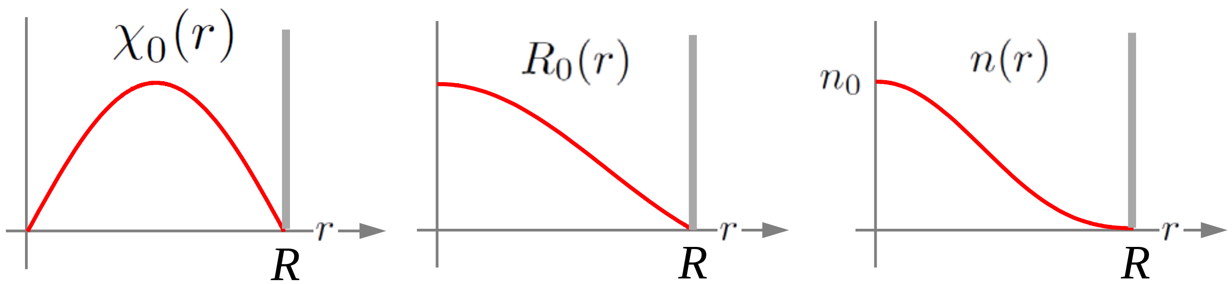
significance of *s*-wave scattering length

$$\boxed{\delta E = \frac{\hbar^2}{2m}\left(k_n'^2 - k_n^2\right) \simeq n^2 \frac{\hbar^2}{m}\frac{\pi^2}{R^3} a}$$

significance of *s*-wave scattering length

$$\boxed{\delta E = \frac{\hbar^2}{2m}\left(k_n'^2 - k_n^2\right) \simeq n^2 \frac{\hbar^2}{m}\frac{\pi^2}{R^3} a}\quad \begin{cases} \text{repulsive for } a > 0 \\ \text{attractive for } a < 0 \end{cases}$$

ground state ($n = 0$):



$$1 = \int |\psi_0(\mathbf{r})|^2\, d\mathbf{r} = \int n(\mathbf{r}) d\mathbf{r} = n_0 \underbrace{\int \frac{n(\mathbf{r})}{n_0} d\mathbf{r}}_{V_e}$$

$$V_e \longrightarrow V_e \equiv \frac{N}{n_0} = \frac{2}{\pi} R^3$$

$$\boxed{\delta E = \frac{4\pi\hbar^2}{m}\frac{a}{V_e}}$$

pedestrian view on chemical potential

$$\delta E = \frac{4\pi\hbar^2}{m}\frac{a}{V_e} \quad \longrightarrow \quad E = \delta E\frac{N(N-1)}{2} \quad \longrightarrow \quad \boxed{E \simeq \frac{2\pi\hbar^2}{m}a\frac{N^2}{V_e}}$$

Chemical potential at $T = 0$:

$$\mu = \frac{\partial E}{\partial N} = \frac{4\pi\hbar^2}{m}a\, n_0$$

$$\boxed{\mu = g_0 n_0} \qquad \boxed{g_0 = \frac{4\pi\hbar^2}{m}a}$$

$$\boxed{\mathcal{V}(\mathbf{r}_1 - \mathbf{r}_2) = g_0\delta(\mathbf{r}_1 - \mathbf{r}_2)}$$

Relative motion of interacting particles II

flat-bottom potential

effective range - r_e

effective range r_e

We now analyze the *energy dependence* of the phase shift

$$\eta_0(k) = \underbrace{-kr_0}_{\substack{\alpha \\ \text{“regular”}}} + \underbrace{\arctan \frac{k}{K_+ \cot K_+ r_0}}_{\substack{\beta \\ \text{“resonant”}}}$$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$$

$$\tan x = x + \tfrac{1}{3}x^3 + \cdots = x(1 + \tfrac{1}{3}x^2 + \cdots)$$

evaluate phase shift:

$$kr_0 \cot \eta_0 = \frac{K_+ r_0 \cot K_+ r_0 + k^2 r_0^2 + \cdots}{1 - \left(1 + \frac{1}{3}k^2 r_0^2 + \cdots\right) K_+ r_0 \cot K_+ r_0}$$

$$K_+^2 = k^2 + \kappa_0^2 \longrightarrow K_+ r_0 = \kappa_0 r_0 [1 + k^2/\kappa_0^2]^{1/2} = \gamma + \tfrac{1}{2} k^2 r_0^2/\gamma + \cdots$$

effective range r_e

$$k r_0 \cot \eta_0 = -\frac{1}{1-\tan\gamma/\gamma} + \tfrac{1}{2}k^2 r_0^2 \left(1 - \frac{3\,(1-\tan\gamma/\gamma)+\gamma^2}{3\gamma^2\,(1-\tan\gamma/\gamma)^2}\right) + \cdots$$

$$\boxed{a = r_0\,(1-\tan\gamma/\gamma)}$$

r_e : measure for energy dependence of the phase shift

Effective range expansion:

$$\boxed{k\cot\eta_0 = -\frac{1}{a} + \tfrac{1}{2}k^2 r_e + \cdots} \rightarrow \boxed{k\cot\eta_0 = -\frac{1}{a(k)}}$$

Effective range: $r_e = r_0\left(1 - \dfrac{3ar_0 + \gamma^2 r_0^2}{3\gamma^2 a^2}\right)$

Two expansions to order k^2:

$$a \neq 0 \qquad k\cot\eta_0 = -\frac{1}{a} + \tfrac{1}{2}k^2 r_0\left(1 - \frac{3ar_0+\gamma^2 r_0^2}{3\gamma^2 a^2}\right) + \cdots$$

$$a \ll r_0 \qquad \frac{1}{k\cot\eta_0} = -a + \tfrac{1}{6}k^2 r_0^3[1 - 3\,(a/r_0)^2 + (3/\gamma^2)\,(a/r_0)] + \cdots$$

effective range r_e – special cases

$$a \neq 0 \qquad k\cot\eta_0 = -\frac{1}{a} + \tfrac{1}{2}k^2 \underbrace{r_0\left(1 - \frac{3ar_0 + \gamma^2 r_0^2}{3\gamma^2 a^2}\right)}_{r_e} + \cdots$$

positive

$a = \pm r_0$ (regular) $\qquad r_e = r_0\left(2/3 \mp 1/\gamma^2\right) \simeq \frac{2}{3}r_0$

$$k\cot\eta_0 = \mp\frac{1}{r_0} + \tfrac{1}{3}\cancel{k^2}r_0 + \cdots$$

$kr_0 \ll 1$ ⟶ k dependence unimportant

positive

$|a| \gg r_0$ (anomalously large) $\qquad r_e \simeq r_0$

$$k\cot\eta_0 = \cancel{-\frac{1}{a}} + \tfrac{1}{2}k^2 r_0 + \cdots$$

⟶ strong k dependence of $a(k)$

effective range r_e – special cases

$$a \ll r_0 \qquad \frac{1}{k\cot\eta_0} = \cancel{-a} + \tfrac{1}{6}k^2 r_0^3 [1 - 3\,(a/r_0)^2 + (3/\gamma^2)\,(a/r_0)] + \cdots$$

$a = 0$ (anomalously small)

$$k\cot\eta_0 = -\frac{1}{a(k)} = \frac{6}{k^2 r_0^3} + \cdots \qquad a(k) = -\tfrac{1}{6}k^2 r_0^3$$

$kr_0 \ll 1$ $\longrightarrow$ k dependence large but a always small

Conclusions for ‘open channel’ potentials:
effective range always positive - important only for $|a| \gg r_0$

examples

ultracold atom collisions:

^{133}Cs |4,4>	$a = 2400\, a_0$	$r_0 = 100\, a_0$
^{85}Rb |3,3>	$a = -369\, a_0$	$r_0 = 83\, a_0$
^{88}Sr	$a = -2\, a_0$	
^{1}H |1,1>	$a = 1.22\, a_0$	$r_e = 348\, a_0$

2 hadron collisions at MeV energies:

s-wave regime ($kr_0 << 1$):

proton (uud) I=1/2

neutron (udd) I=1/2

r_0 is 6 orders of magnitude smaller

k can be 6 orders of magnitude larger

deuteron	I=1	bound state	$a > 0$	$a = 5.41$ fm $r_e = 1.75$ fm
	I=0	virtual state	$a < 0$	$a = -2.38$ fm $r_e = 2.67$ fm

Relative motion of interacting particles II

1. We explored the scattering length
2. We studied its dependence on the well parameter
3. We demonstrated elastic scattering with Rb-87
4. We found resonant enhancement near the bound state threshold
5. We found universal behavior near the bound state threshold
6. We defined halo states and zero-range potentials
7. We analyzed the energy dependence of the scattering length
8. We introduced the effective range r_e
9. We analyzed when the effective range is important
10. We discussed anomalously large and small scattering lengths
11. We discussed some examples