Elastic Collision Phenomena in Quantum Gases

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outline

Course of five lectures on collision phenomena in quantum gases

1. Relative motion of interacting particles I
   - model potentials: range, phase shift, scattering length
2. Relative motion of interacting particles II
   - model potentials: effective range and s-wave resonance
   - generalization to arbitrary short-range potentials
3. Scattering of interacting particles
   - scattering amplitude and cross section
   - distinguishable versus identical particles
4. Scattering of particles with internal structure (atoms)
5. Interaction tuning with magnetic Feshbach resonance

Many details given in the lecture notes for Les Houches-2019

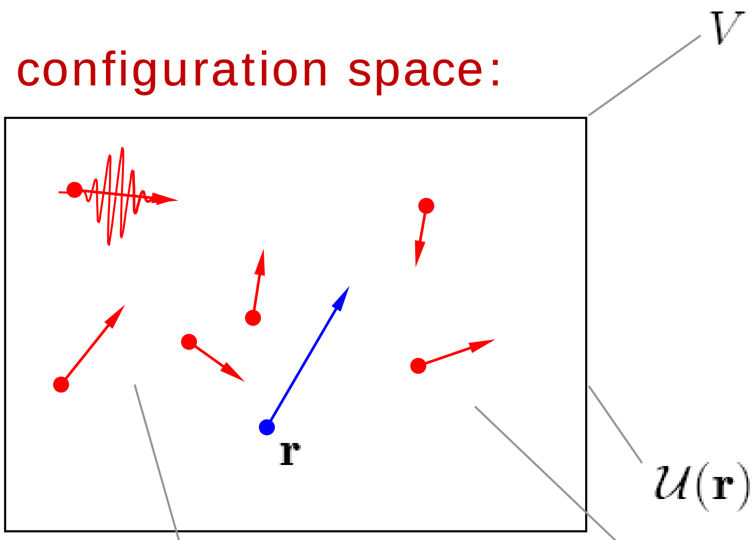
Journal club suggestion: Polaron problem

Seminal paper (theory): P.W. Anderson, INFRARED CATASTROPHE IN FERMI GASES WITH LOCAL SCATTERING POTENTIALS, Phys. Rev. Lett. 18, 1049 (1967)

Polaron formation (expt): M. Cetina et al., ULTRAFAST MANY-BODY INTERFEROMETRY OF IMPURITIES COUPLED TO A FERMI SEA, Science 354 96 (2016)

Gas phase and quantum resolution

configuration space:



density: $n(\mathbf{r})$ temperature: T

classical particles:

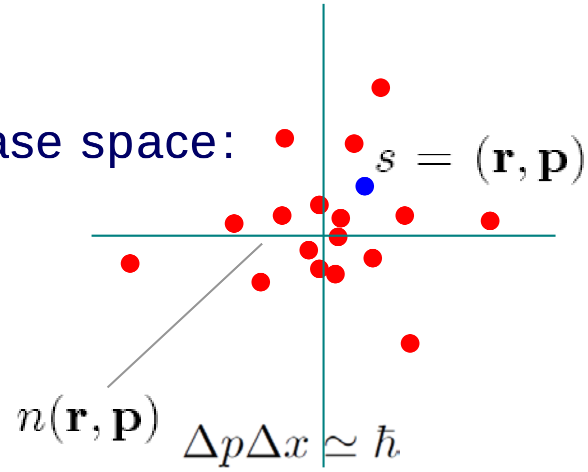
position: $\mathbf{r}$

momentum: $\mathbf{p} = m\mathbf{v}$

$s = (\mathbf{r}, \mathbf{p})$ kinetic state

Gas: $\{\mathbf{r}_i, \mathbf{p}_i\}$ $i \in \{1, \cdots N\}$

phase space:



quantum mechanical description ← (quantum resolution limit)

Lectures on quantum gases

Lecture 1

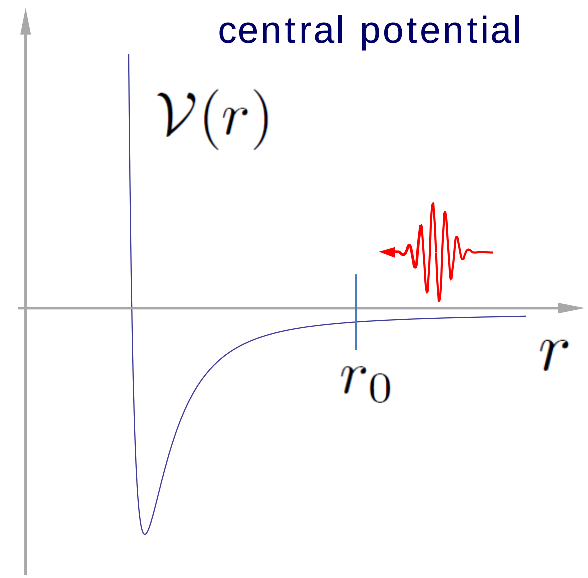
Relative motion of interacting particles I

Characteristic lengths and quantum regimes

interaction range: $r_0 \ll n_0^{-1/3} \ll V^{1/3}$

interatomic spacing: → $n_0^{-1/3}$

system size: → $V^{1/3}$



definition quantum regimes:

thermal wavelength: $\Lambda \equiv \sqrt{\dfrac{2\pi\hbar^2}{mk_BT}}$

$k \sim 1/\Lambda$

$\Lambda \ll r_0 \rightarrow kr_0 \gg 1$ quasi-classical collisions

$\Lambda \gg r_0 \rightarrow kr_0 \ll 1$ ultracold collisions

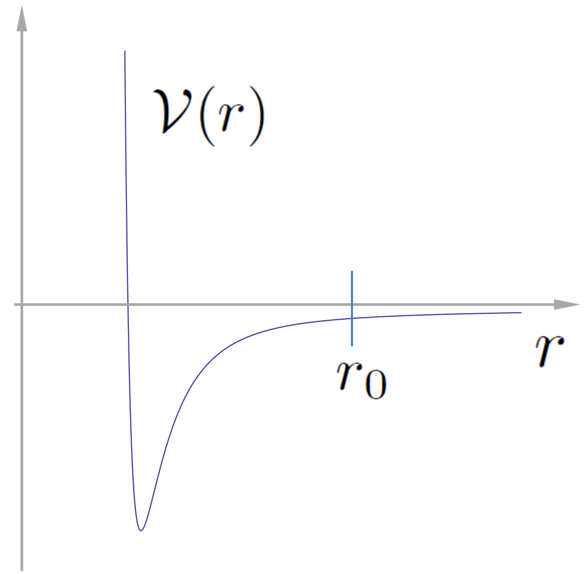
quantum gas: $kr_0 \ll 1$ (degenerate for $n_0\Lambda^3 \gg 1$)

short-range interactions – collisional regimes

only binary interactions: $nr_0^3 \ll 1$ *dilute*

nearly ideal
weakly interacting } $na^3 \ll 1$

s-wave scattering length



cross section: $\sigma \simeq 4\pi a^2$

interaction ranges: r_0, a, r_e, R^*

collision rate: $\tau_c^{-1} = n\bar{v}_r\sigma$

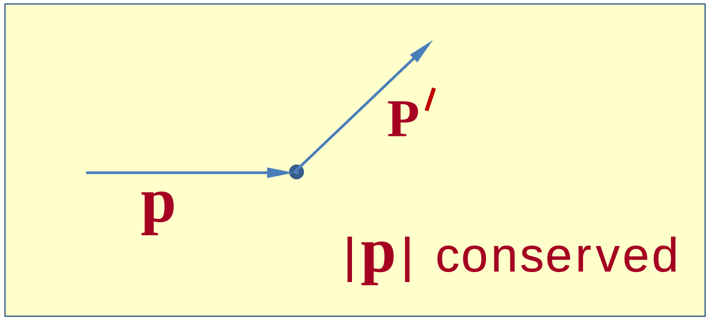
mean free path: $\ell = 1/n\sigma$ → $\ell \gg V^{1/3}$ *collisionless*
$\ell \ll V^{1/3}$ *hydrodynamic (collisional)*

kinematics of binary collision

CM and REL coordinates:

relative position: $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

relative velocity: $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$



closed system: conserved quantities E and $\mathbf{P}$

no external fields
(kinetic momentum)

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 \doteq m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 = \overbrace{(m_1 + m_2)}^{M}\frac{d}{dt}\overbrace{\frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}}^{\mathbf{R}} = M\,\dot{\mathbf{R}}$$

$\mathbf{P} = M\,\dot{\mathbf{R}}$ conserved

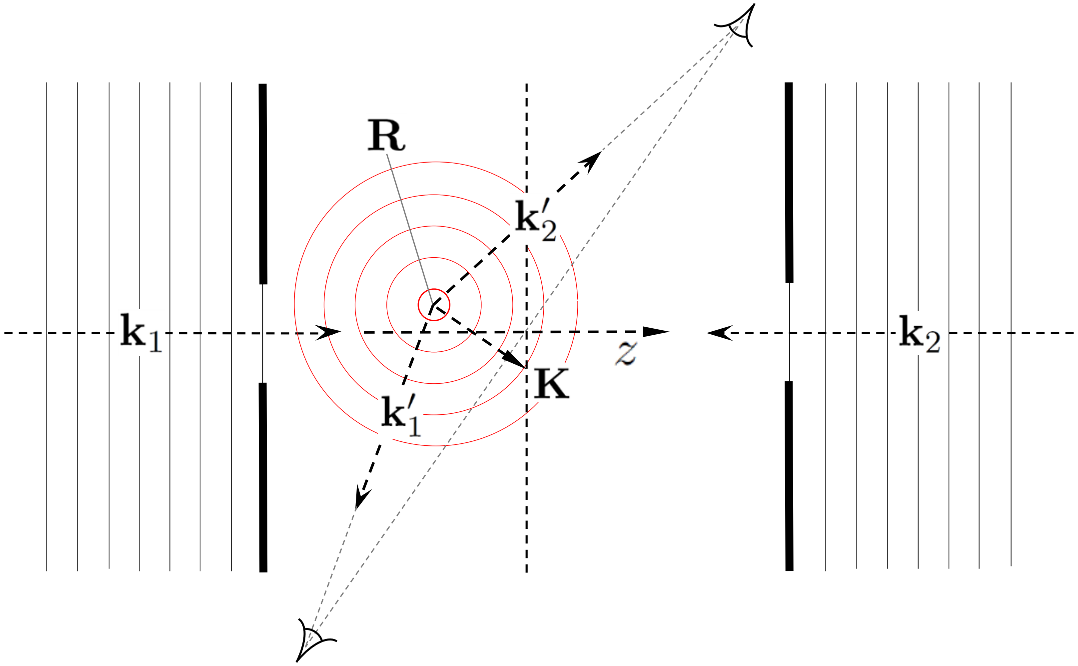
$$E = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} = \frac{P^2}{2M} + \frac{p^2}{2\mu}$$

$\rightarrow$ $\dfrac{p^2}{2\mu}$ conserved

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

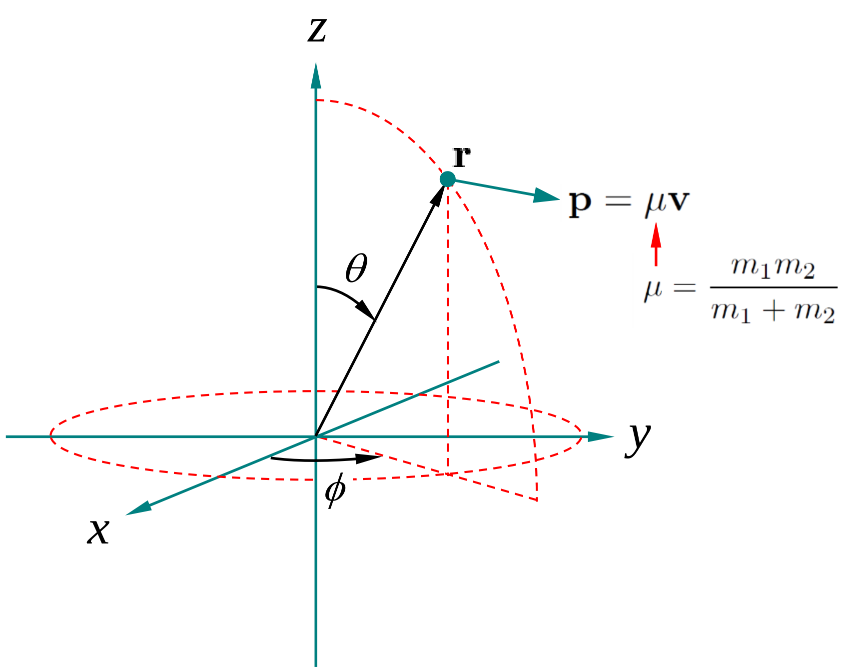
relative momentum: $\mathbf{p} = \mu\mathbf{v}$

Quantum limitations



experiments diffraction limited

central potential



potential energy

Hamiltonian:

$$H = \frac{\mathbf{p}^2}{2\mu} + \mathcal{V}(\mathbf{r})$$

central potential: $\mathcal{V}(\mathbf{r}) = \mathcal{V}(r)$

$$\mathbf{p}^2 = (\hat{\mathbf{r}} \cdot \mathbf{p})^2 + (\hat{\mathbf{r}} \times \mathbf{p})^2$$

$$p_r = \hat{\mathbf{r}} \cdot \mathbf{p} \qquad \mathbf{L} = \mathbf{r} \times \mathbf{p}$$

spherical symmetry allows separation of radial and angular motion:

$$H = \frac{1}{2\mu}\left(p_r^2 + \frac{\mathbf{L}^2}{r^2}\right) + \mathcal{V}(r)$$

$r \neq 0$

check solution
for regularity in the origin

Schrödinger equation for the relative motion

$$\left[\frac{1}{2\mu}\left(p_r^2 + \frac{\mathbf{L}^2}{r^2}\right) + \mathcal{V}(r)\right]\psi(r,\theta,\phi) = E\psi(r,\theta,\phi)$$

$\mathbf{L}^2, L_z$ commute with each other and with r and p_r

separation of variables: $\psi = R_l(r)Y_l^m(\theta,\phi)$

$$\mathbf{L}^2\, Y_l^m(\theta,\phi) = l(l+1)\hbar^2\, Y_l^m(\theta,\phi)$$
$$L_z\, Y_l^m(\theta,\phi) = m\hbar\, Y_l^m(\theta,\phi).$$

$$\left[\frac{1}{2\mu}\left(p_r^2 + \frac{l(l+1)\hbar^2}{r^2}\right) + \mathcal{V}(r)\right] R_l(r)\cancel{Y_l^m(\theta,\phi)} = E R_l(r)\cancel{Y_l^m(\theta,\phi)}$$

radial wave equation:

$$\frac{2\mu}{\hbar^2}\left[\frac{\hbar^2}{2\mu}\left(-\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr}\right) + \underbrace{\frac{l(l+1)\hbar^2}{2\mu r^2} + \mathcal{V}(r)}_{\mathcal{V}_{\text{eff}}(r)}\right] R_l(r) = E R_l(r)$$

radial wave equation

$$R_l'' + \frac{2}{r} R_l' + \left[\varepsilon - U(r) - \frac{l(l+1)}{r^2} \right] R_l = 0$$

we changed to wavenumber notation

$$U(r) = 2\mu \mathcal{V}(r)/\hbar^2 \qquad \varepsilon = 2\mu E/\hbar^2 \begin{cases} \varepsilon = k^2 \\ \varepsilon = -\kappa^2 \end{cases}$$

continuum states $(\varepsilon > 0)$

bound states $(\varepsilon < 0)$

introduce reduced wavefunction: $\chi_l(r) = r R_l(r)$

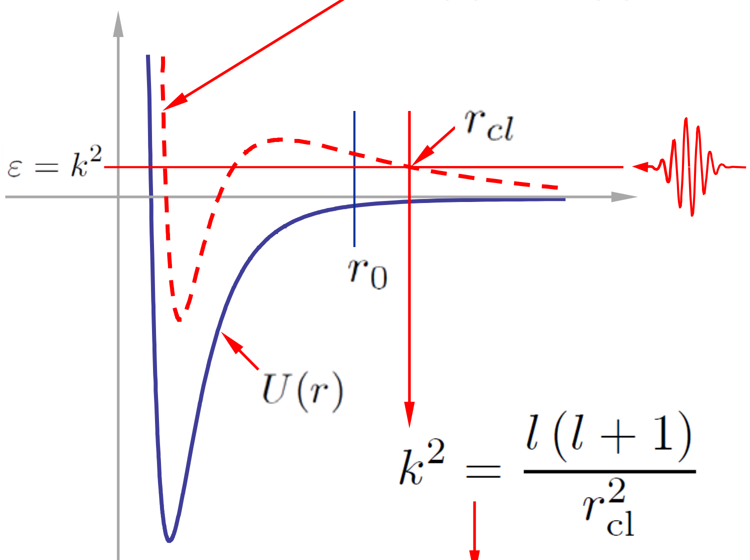
1D Schrödinger equation:
radial wave equation:

$$\chi_l'' + [\varepsilon - U(r) - \frac{l(l+1)}{r^2}]\chi_l = 0$$

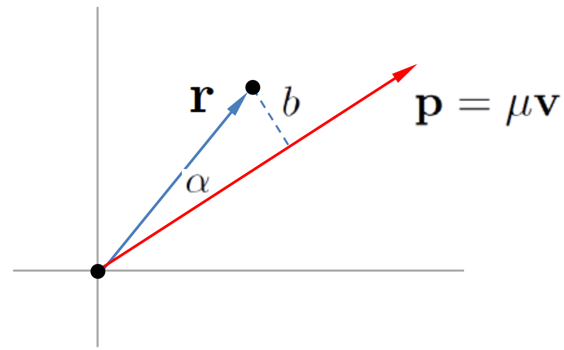
$$\frac{2\mu}{\hbar^2}\left[\frac{\hbar^2}{2\mu}\left(-\frac{d^2}{dr^2} \right.\right. \ldots R_l(r)$$

s-wave regime

$$U_{\text{eff}}(r) = U(r) + \frac{l(l+1)}{r^2}$$



$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \qquad L = p\,\overbrace{r \sin\alpha}^{b}$$



$$k^2 = \frac{l\,(l+1)}{r_{\text{cl}}^2} \qquad l \neq 0$$

$$k r_0 = \sqrt{l(l+1)}\,\frac{r_0}{r_{cl}} \ll 1 \quad \rightarrow \quad \boxed{r_0 \ll r_{cl}}$$

<u>conclusion:</u> for $k r_0 \ll 1$ no collisions with l > 0

only s-wave collisions

(exception: quasi-bound states in continuum/shape resonances)

free particle motion for $l = 0$

$$\chi_l'' + \left[k^2 - U(r) - \frac{l(l+1)}{r^2}\right]\chi_l = 0$$

$l = 0$ and $U(r) = 0$

↓

$$\chi_0'' + k^2\chi_0 = 0$$

General solution:

$$R_0 = c_0 \frac{\sin(kr + \eta_0)}{kr}$$

regular only for $\eta_0 = 0$

↓

Conclusion: free particle – no phase shift

Relative motion of interacting particles I

1. We defined what we mean by ultracold – characteristic lengths
2. Short-range interactions - collisional regimes
3. Separation into CM and REL coordinates
4. We derived the radial wave equation
5. We defined the s-wave regime
6. We identified the phase shift as the central quantity of interest